SHORTER NOTES

The purpose of this department is to publish very short papers of an unusually elegant and polished character, for which there is no other outlet.

A SHORT PROOF OF A LEMMA OF G. R. MACLANE

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Let f be holomorphic and nonconstant in $D=\{|z|<1\}$. The level set $L(\lambda)$ of f is defined as $\{z: |f(z)|=\lambda, \lambda \geqq 0\}$. In [1] the following lemma [1, p. 10] is quite fundamental.

LEMMA. *If f is holomorphic, bounded, and nonconstant in D, then for no λ can $L(\lambda)$ contain a sequence of disjoint arcs converging to a nondegenerate arc of $C=\{|z|=1\}$ (i.e., $L(\lambda)$ "ends at points" of C).*

MacLane's proof of this lemma is rather involved and computational.

The object of this note is to give a very short geometric proof of the above lemma.

PROOF. Assume to the contrary that for some λ there exists a sequence of disjoint arcs $\gamma_n \subset L(\lambda)$ converging to a nondegenerate arc $\gamma \subset C$. Let S denote the Riemannian image of f. Since f is an open map and $|f|=\lambda$ on γ_n, we can find a sequence $\{z_n\}$ of points in D which satisfy:

(i) $z_n \to e^{i\alpha}$, where $e^{i\alpha}$ is the midpoint of γ,
(ii) $|f(z_n)|>\lambda$,
(iii) $f(z_n) \to a$, where $|a|=\lambda$,
(iv) there are no branch points of S above any of the rays

$$R_n = \{w: |w| \geqq |f(z_n)|, \arg w = \arg f(z_n)\}.$$

Let p_n be any point in S over $f(z_n)$. There exists a maximal half open segment $[f(z_n), b_n) \subset R_n$ that can be lifted into S with starting point p_n. Call the lifted curve $\tilde{R}_n$ and let Γ_n be its preimage back in D. (Note that $\Gamma_n \cap \gamma_k$ must be empty for all k and that on Γ_n the function f must have the asymptotic value b_n.) Hence we can find a sequence $\{\tau_k\}$ of Jordan arcs and a nondegenerate subarc $\tilde{\gamma} \subset \gamma$ such that each τ_k is contained in some Γ_n and $\tau_k \to \tilde{\gamma}$ (this follows from the way we chose $e^{i\alpha}$).

Received by the editors July 1, 1968.

Now consider any point $e^{i\theta} \in \operatorname{Int} \tilde{\gamma}$ at which the radial limit $f(e^{i\theta})$ exists. Since the radius to $e^{i\theta}$ must intersect infinitely many of the γ_n and infinitely many of the τ_k, it follows that $|f(e^{i\theta})| = \lambda$ and $\arg f(e^{i\theta}) = \arg a$. Hence by the Riesz uniqueness theorem $f \equiv a$. Contradiction.

REFERENCE

1. G. R. MacLane, *Asymptotic values of holomorphic functions*, Rice Univ. Studies, 49, No. 1 (1963).

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