A REMARK ON A COMPARISON THEOREM
OF SWANSON

KURT KREITH

In [1] C. A. Swanson proves a comparison theorem for sufficiently regular, second order elliptic equations of the form

\[ L^*u = \sum_{i,j=1}^{n} D_i(a_{ij}D_ju) + 2 \sum_i b_i D_iu + c^*u = 0, \]

\[ Lv = \sum_{i,j=1}^{n} D_i(a_{ij}D_jv) + 2 \sum_i b_i D_iv + cv = 0, \]

defined in a domain \( R \) with piecewise continuous unit normal on the boundary \( B \). Given that \( L \) is a strict Sturmian majorant of \( L^* \) and that there exists a nontrivial solution of (1) satisfying \( u = 0 \) on \( B \), Swanson shows that every solution of (2) has a zero in \( \bar{R} \). This result is not "strong" in the sense of [2] where it is shown that under similar hypotheses in the selfadjoint case, every solution of (2) must vanish in the interior of \( R \).

The purpose of this note is to point out that if \( B \) is of bounded curvature, then one can use the method of [1] to arrive at this stronger conclusion even in the nonselfadjoint case. Specifically, if it can be shown that every solution of (2) which is not zero in \( R \) and vanishes at a point \( x_0 \in B \) must satisfy \( \frac{d}{dx} (x_0) \neq 0 \), then it is clear from the proof that the Lemma of [1] can be altered to read: "Suppose \( g \) satisfies \( g \det(a_{ij}) > -\sum_{i=1}^{n} b_i B_i \). If there exists \( u \in C^1 \) not identically zero such that \( J[u] \leq 0 \), then every solution \( v \) of \( Lv = 0 \) vanishes at some point of \( R \)." A strong version of Swanson's comparison theorem follows readily from this change, and in the case of ordinary differential equations (i.e. \( n = 1 \)) this fact is observed in [1].

If \( c < 0 \) near \( B \) and \( B \) is of bounded curvature, then it follows from the Hopf maximum principle [3] that \( (\partial v/\partial v)(x_0) \neq 0 \) whenever \( v(x_0) = 0 \), \( x_0 \in B \). However, even if \( c \) is merely bounded, the same conclusion can be derived.

To see this we assume \( v < 0 \) in \( R \) and \( v(x_0) = 0 \) for some \( x_0 \in B \). Without loss of generality we may assume that \( B \) is tangent to the plane \( x_1 = b \) and that the exterior normal to \( B \) at \( x_0 \) is in the positive \( x_1 \)-direction. It is known (see [4, p. 73]) that for \( (b - a) \) sufficiently small there exist positive constants \( \alpha, \beta \) such that

Received by the editors November 2, 1967.

549
\[ w(x) = 1 - \beta e^{(x_1 - a)} > 0 \quad \text{for } a \leq x_1 \leq b; \]
\[ Lw \leq 0 \quad \text{for } a \leq x_1 \leq b. \]

Furthermore, a direct computation (see [4, p. 72]) shows that the Hopf maximum principle applies to \( v/w \) in the intersection of the slab \( a < x_1 < b \) with \( R \). Since \( v/w \) has a nonnegative maximum at \( x_0 \), it follows that at \( x_0 \)

\[ 0 < \frac{\partial}{\partial v} \left( \frac{v}{w} \right) = \frac{w(\partial v/\partial x_1) - v(\partial w/\partial x_1)}{w^2} = \frac{1}{w} \frac{\partial v}{\partial v}. \]

Therefore \( \partial v/\partial v > 0 \) at \( x_0 \) and the strong comparison theorem follows.

These remarks can also be used to strengthen some of the conclusions of [5] for comparison theorems in unbounded domains.

**Bibliography**


**University of California, Davis**