# A REMARK ON A COMPARISON THEOREM OF SWANSON

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In [1] C. A. Swanson proves a comparison theorem for sufficiently regular, second order elliptic equations of the form

$$(1)\qquad L^*u \equiv \sum_{i,j=1}^{n} D_i(\overset{*}{a}_{ij}D_ju) + 2\sum_i \overset{*}{b}_i D_iu + c^*u = 0,$$

$$(2)\qquad Lv \equiv \sum_{i,j=1}^{n} D_i(a_{ij}D_jv) + 2\sum_i b_iD_iv + cv = 0,$$

defined in a domain $R$ with piecewise continuous unit normal on the boundary $B$. Given that $L$ is a strict Sturmian majorant of $L^*$ and that there exists a nontrivial solution of (1) satisfying $u=0$ on $B$, Swanson shows that every solution of (2) has a zero in $\bar{R}$. This result is not "strong" in the sense of [2] where it is shown that under similar hypotheses in the selfadjoint case, every solution of (2) must vanish in the interior of $R$.

The purpose of this note is to point out that if $B$ is of bounded curvature, then one can use the method of [1] to arrive at this stronger conclusion even in the nonselfadjoint case. Specifically, if it can be shown that every solution of (2) which is not zero in $R$ and vanishes at a point $\mathbf{x}_0\in B$ must satisfy $(\partial v/\partial\nu)(\mathbf{x}_0)\neq 0$, then it is clear from the proof that the Lemma of [1] can be altered to read: "Suppose $g$ satisfies $g\det(a_{ij}) > -\sum_{i=1}^n b_iB_i$. If there exists $u\in\mathfrak{D}$ not identically zero such that $J[u]\leqq 0$, then every solution $v$ of $Lv=0$ vanishes at some point of $R$." A strong version of Swanson's comparison theorem follows readily from this change, and in the case of ordinary differential equations (i.e. $n=1$) this fact is observed in [1].

If $c\leqq 0$ near $B$ and $B$ is of bounded curvature, then it follows from the Hopf maximum principle [3] that $(\partial v/\partial\nu)(\mathbf{x}_0)\neq 0$ whenever $v(\mathbf{x}_0)=0$, $\mathbf{x}_0\in B$. However, even if $c$ is merely bounded, the same conclusion can be derived.

To see this we assume $v<0$ in $R$ and $v(\mathbf{x}_0)=0$ for some $\mathbf{x}_0\in B$. Without loss of generality we may assume that $B$ is tangent to the plane $x_1=b$ and that the exterior normal to $B$ at $\mathbf{x}_0$ is in the positive $x_1$-direction. It is known (see [4, p. 73]) that for $(b-a)$ sufficiently small there exist positive constants $\alpha$, $\beta$ such that

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$$w(\mathbf{x}) \equiv 1 - \beta e^{\alpha(x_1 - a)} > 0 \qquad \text{for } a \leqq x_1 \leqq b;$$
$$Lw \leqq 0 \qquad \text{for } a \leqq x_1 \leqq b.$$

Furthermore a direct computation (see [4, p. 72]) shows that the Hopf maximum principle applies to $v/w$ in the intersection of the slab $a < x_1 < b$ with $R$. Since $v/w$ has a nonnegative maximum at $\mathbf{x}_0$, it follows that at $\mathbf{x}_0$

$$0 < \frac{\partial}{\partial \nu}\left(\frac{v}{w}\right) = \frac{w(\partial v/\partial x_1) - v(\partial w/\partial x_1)}{w^2} = \frac{1}{w}\,\frac{\partial v}{\partial \nu}\,.$$

Therefore $\partial v/\partial \nu > 0$ at $\mathbf{x}_0$ and the strong comparison theorem follows.

These remarks can also be used to strengthen some of the conclusions of [5] for comparison theorems in unbounded domains.

## Bibliography


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