CONCERNING POLYNOMIALS ON THE UNIT INTERVAL

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Let $f(x)=\sum_{\nu=0}^{n} a_\nu x^\nu$ be a polynomial of degree n. Given its size on the interval $[-1, 1]$ the general problem is to determine how large $|f^{(k)}(x)|$ can be on the same interval. If we know $\|f\|_\infty=\max_{[-1,1]}|f(x)|$ as a measure of the size then a classical theorem of W. Markoff [2] states that

$$(1)\qquad \|f^{(k)}\|_\infty \leqq \frac{n^2(n^2-1^2)(n^2-2^2)\cdots(n^2-(k-1)^2)}{1\cdot 3\cdot 5\cdots(2k-1)}\|f\|_\infty.$$

Here we assume that $\|f\|_2=(\int_{-1}^{+1}|f(x)|^2dx)^{1/2}$ is given as a measure of the size of f and we obtain a sharp bound for $|f^{(k)}(x)|$ as well as for the coefficients a_k. Our result may be stated as follows.

THEOREM. *Let $f(x)=\sum_{\nu=0}^{n} a_\nu x^\nu$ be a polynomial of degree n, then for any fixed k, $0\leqq k\leqq n$,*

$$(2)\qquad \|f^{(k)}\|_\infty \leqq 1\cdot 3\cdot 5\cdots(2k-1)\cdot\frac{n+k+1}{(2(2k+1))^{1/2}}\binom{n+k}{n-k}\|f\|_2$$

and

$$(3)\qquad |a_k| \leqq \frac{1\cdot 3\cdot 5\cdots(2k-1)}{k!}(k+\tfrac{1}{2})^{1/2}\binom{[(n-k)/2]+k+\frac{1}{2}}{[(n-k)/2]}\|f\|_2,$$

where the symbol $[x]$ denotes as usual the integral part of x.

The above inequalities are best possible. Equality in (2) *holds only for constant multiples of the polynomial*

$$(4)\qquad \sum_{\mu=0}^{n-k}(2\mu+2k+1)\binom{2k+\mu}{\mu}P_{k+\mu}(x).$$

Besides, the supremum is attained only at the end points of $[-1, 1]$. *Equality in* (3) *is attained only for the constant multiples of the polynomial*

$$(5)\qquad \sum_{\mu=0}^{[(n-k)/2]}(-1)^\mu(4\mu+2k+1)\binom{k+\mu-\frac{1}{2}}{\mu}P_{k+2\mu}(x).$$

The symbol $P_\nu(x)$ in (4) *and* (5) *denotes the Legendre polynomial of degree ν.*

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For the proof of the theorem we need a lemma.

LEMMA. *If $P_n(x)$ denotes the Legendre polynomial of degree n, then for $0<\theta<\pi$, $k\geqq 0$ we have*

$$P_n^{(k)}(\cos\theta) = 1\cdot 3\cdot 5\cdots(2k-1)\sum_{\nu=0}^{n} D(k, n-\nu;\theta)P_\nu(\cos\theta), \tag{6}$$

where

$$D(k,\mu;\theta) = \sum_{\nu_1+\cdots+\nu_k=\mu} \frac{\sin\nu_1\theta\cdots\sin\nu_k\theta}{\sin^k\theta}. \tag{7}$$

We also have

$$P_n^{(k)}(1) = 1\cdot 3\cdot 5\cdots(2k-1)\binom{n+k}{n-k}, \tag{8}$$

$$P_n^{(k)}(-1) = 1\cdot 3\cdot 5\cdots(2k-1)\cdot(-1)^{n-k}\binom{n+k}{n-k}, \tag{9}$$

$$P_n^{(k)}(0) = 1\cdot 3\cdot 5\cdots(2k-1)\cdot(-1)^{(n-k)/2}\left(\begin{matrix}\frac{n+k-1}{2}\\ \frac{n-k}{2}\end{matrix}\right) \quad \textit{if } n\equiv k \pmod 2, \tag{10}$$

$$= 0 \qquad \textit{if } n\not\equiv k \pmod 2.$$

PROOF OF THE LEMMA. It is well known (see for example [1, p. 365]) that

$$(1-2xz+z^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)z^n \qquad \text{for } -1<x<1,\ |z|<1. \tag{11}$$

Differentiating both sides of (11) k times with respect to x, we get

$$\frac{1\cdot 3\cdot 5\cdots(2k-1)z^k}{(1-2xz+z^2)^k}\cdot\frac{1}{\sqrt{1-2xz+z^2}} = \sum_{n=0}^{\infty} P_n^{(k)}(x)z^n$$

$$\text{for } -1<x<1,\ |z|<1.$$

In other words, for $x=\cos\theta$, $0<\theta<\pi$, we have

$$\frac{1\cdot 3\cdot 5\cdots(2k-1)z^k}{[(z-e^{i\theta})(z-e^{-i\theta})]^k}\cdot\frac{1}{\sqrt{1-2z\cos\theta+z^2}} = \sum_{n=0}^{\infty} P_n^{(k)}(\cos\theta)z^n. \tag{12}$$

But

$$\frac{z}{(z-e^{i\theta})(z-e^{-i\theta})} = \sum_{\nu=0}^{\infty}\left(\frac{\sin \nu\theta}{\sin\theta}\right)z^{\nu},$$

so that

$$\frac{z^k}{[(z-e^{i\theta})(z-e^{-i\theta})]^k} = \left\{\sum_{\nu=0}^{\infty}\left(\frac{\sin \nu\theta}{\sin\theta}\right)z^{\nu}\right\}^k = \sum_{\mu=0}^{\infty} D(k,\mu;\theta)z^{\mu}. \tag{13}$$

Substituting (13) in (12) we deduce that

$$1\cdot 3\cdot 5\cdots(2k-1)\left(\sum_{\mu=0}^{\infty} D(k,\mu;\theta)z^{\mu}\right)\left(\sum_{\nu=0}^{\infty} P_{\nu}(\cos\theta)z^{\nu}\right) = \sum_{n=0}^{\infty} P_n^{(k)}(\cos\theta)z^n$$

that is

$$1\cdot 3\cdot 5\cdots(2k-1)\sum_{n=0}^{\infty}\left(\sum_{\nu=0}^{n} D(k,n-\nu;\theta)P_{\nu}(\cos\theta)\right)z^n = \sum_{n=0}^{\infty} P_n^{(k)}(\cos\theta)z^n$$

and (6) follows from equating the coefficients of z^n on both sides of this last inequality. Equalities (8), (9), (10) follow easily from standard generating functions.

PROOF OF THE THEOREM. Let

$$\phi_{\nu}(x) = (\nu+\tfrac{1}{2})^{1/2}P_{\nu}(x)$$

denote the "normalized" Legendre polynomial of degree ν. Then the given polynomial $f(x)$ can be expressed uniquely in the form

$$f(x) = \sum_{\nu=0}^{n} \alpha_{\nu}\phi_{\nu}(x) \tag{14}$$

where

$$\|f\|_2 = \left(\sum_{\nu=0}^{n} \alpha_{\nu}^2\right)^{1/2}. \tag{15}$$

Hence, for a fixed x_0 $(=\cos\theta_0)$ in $[-1, 1]$, we get

$$|f^{(k)}(x_0)| = \left|\sum_{\nu=0}^{n} \alpha_\nu \phi_\nu^{(k)}(\cos\theta_0)\right|$$

(16)
$$\leqq \left[\sum_{\nu=0}^{n} \{\phi_\nu^{(k)}(\cos\theta_0)\}^2\right]^{1/2} \left(\int_{-1}^{+1} |f(x)|^2 dx\right)^{1/2},$$

by Schwarz's inequality

$$= \left[\sum_{\nu=0}^{n} (\nu+\tfrac{1}{2})\{P_\nu^{(k)}(\cos\theta_0)\}^2\right]^{1/2} \|f\|_2.$$

Equality is attained for and only for the constant multiples of

(16′)
$$\sum_{\nu=0}^{n} \phi_\nu^{(k)}(\cos\theta_0)\phi_\nu(x).$$

Now, if $0<\theta_0<\pi$, then

$$|D(k,\mu;\theta_0)| = \left|\sum_{\nu_1+\cdots+\nu_k=\mu} \frac{\sin\nu_1\theta_0 \cdots \sin\nu_k\theta_0}{\sin^k\theta_0}\right|$$

$$\leqq \sum_{\nu_1+\cdots+\nu_k=\mu} \left|\frac{\sin\nu_1\theta_0}{\sin\theta_0}\right| \cdots \left|\frac{\sin\nu_k\theta_0}{\sin\theta_0}\right|$$

$$< \sum_{\nu_1+\cdots+\nu_k=\mu} \nu_1\cdots\nu_k = D(k,\mu;0) = \binom{\mu+k-1}{\mu-k}.$$

Using this strict inequality in (6) we have

$$|P_n^{(k)}(\cos\theta_0)| < |P_n^{(k)}(1)| = 1\cdot 3\cdot 5\cdots(2k-1)\binom{n+k}{n-k}$$

$$= |P_n^{(k)}(-1)|,$$

so that the constant factor in (16) will be maximum at $\theta_0=0$ and $\theta_0=\pi$.

We then conclude that

(17)
$$\|f^{(k)}\|_\infty \leqq \left[\sum_{\nu=0}^{n} (\nu+\tfrac{1}{2})\{P_\nu^{(k)}(\pm 1)\}^2\right]^{1/2} \|f\|_2$$

$$= \frac{1\cdot 3\cdot 5\cdots(2k-1)}{\sqrt{2}} \left[\sum_{\nu=0}^{n} (2\nu+1)\binom{\nu+k}{\nu-k}^2\right]^{1/2} \|f\|_2$$

(18)
$$= \frac{1\cdot 3\cdot 5\cdots(2k-1)}{\sqrt{2}} \cdot \frac{n+k+1}{(2k+1)^{1/2}}\binom{n+k}{n-k}\|f\|_2.$$

To check this summation we proceed as follows. For $n<k$ both (17) and (18) are 0, for $n=k$ it is easy to see that we also have equality and for $n>k$ put

$$A_n = \frac{(n+k+1)^2}{2k+1}\binom{n+k}{n-k}^2;$$

it is easy to check that

$$A_n - A_{n-1} = (2n+1)\binom{n+k}{n-k}^2$$

which is the last term under the summation sign in (17). This completes the proof of (2). We now prove (3).

The inequality (16) gives for $\theta_0=\pi/2$

$$|f^{(k)}(0)| \leqq \left(\sum_{\nu=0}^{n}(\nu+\tfrac{1}{2})\{P_\nu^{(k)}(0)\}^2\right)^{1/2}\|f\|_2. \tag{19}$$

This bound is, by (10), equivalent to

$$\frac{1\cdot 3\cdot 5\cdots(2k-1)}{\sqrt{2}}\left(\sum_{\mu=0}^{[(n-k)/2]}(4\mu+2k+1)\binom{\mu+k-\frac{1}{2}}{\mu}^2\right)^{1/2}\cdot\|f\|_2$$
$$= 1\cdot 3\cdot 5\cdots(2k-1)(k+\tfrac{1}{2})^{1/2}\binom{[(n-k)/2]+k+\frac{1}{2}}{[(n-k)/2]}\cdot\|f\|_2.$$

To check this last equality we need only to show that

$$\sum_{\mu=0}^{m}(4\mu+2k+1)\binom{\mu+k-\frac{1}{2}}{\mu}^2 = (2k+1)\binom{m+k+\frac{1}{2}}{m}^2.$$

For $m=0$ this is true, for $m>0$ put

$$B_m = (2k+1)\binom{m+k+\frac{1}{2}}{m}.$$

It is trivial to verify that

$$B_m - B_{m-1} = (4m+2k+1)\binom{m+k-\frac{1}{2}}{m}^2,$$

whence the result follows.

The extremal polynomials (4) and (5) are obtained on substituting $\theta_0=0$, $\theta_0=\pi$, $\theta_0=\pi/2$ in (16′) and then using (8), (9) and (10).

Our theorem is then completely proved.

Note that

(i) The multiplying factor in (2) is $O(n^{2k+1})$ as $n \to \infty$.
(ii) The multiplying factor in (3) is $O(n^{k+1/2})$ as $n \to \infty$.

We conclude by raising the following question.

What is the smallest number $L = L(k, p, n, e, q)$ such that

$$\|f^{(k)}\|_p \leqq L\|f^{(e)}\|_q, \qquad 1 \leqq p, q \leqq \infty,$$

holds for every polynomial $f(x)$ of degree n?

References

1. P. J. Davis, *Interpolation and approximation*, Blaisdell, New York, 1963.

2. W. Markoff, *Über Polynome, die in einen gegebenen Intervalle möglichst wenig von Null abweichen*, Math. Ann. **77** (1916), 213–258.

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