THE MOD p COHOMOLOGY OF $BO\langle 4k\rangle$

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Let $BO\langle 4k\rangle$ denote the $4k-1$ connected covering of BO, the classifying space for stable vector bundles. $BO\langle 4k\rangle$ is the $4k-1$ connected total space of a fibration $\pi: BO\langle 4k\rangle \to BO$ such that π_*: $\pi_i(BO\langle 4k\rangle) \to \pi_i(BO)$ is an isomorphism for $i \geqq 4k$. We need not, of course, restrict ourselves to integers of the form $4k$, but since we will be concerned in this note only with the mod p behavior, p an odd prime, and since $BO\langle 4k\rangle$ is of the same mod p homotopy type as $BO\langle 4k-i\rangle$, $0 \leqq i < 4$, it suffices to consider only the $4k-1$ connected coverings.

R. Stong [5] computed the cohomology mod 2 of the connected coverings $BO\langle k\rangle$ and $BU\langle k\rangle$ of both BO and BU, and W. Singer [4] the cohomology mod p of $BU\langle k\rangle$. Here, using Singer's results, and methods analogous to and inspired by his, we decomplexify those results to obtain $H^*(BO\langle k\rangle, Z_p)$. Cohomology will, unless otherwise stated, be with coefficients in Z_p, p a fixed odd prime. $F_k \subset H^*(K(Z, k))$ denotes the Hopf subalgebra generated over the Steenrod algebra $\mathcal{A}_p$ by the single element $\beta\mathcal{P}^1\sigma_k \in H^{k+2p-1}(K(Z, k))$. Note F_k is a free commutative Z_p algebra. For any integer n, let $\sigma_p(n)$ denote the sum of the coefficients of the p-adic expansion of n. The main result of this paper is the following theorem.

THEOREM 1. *There exist elements* $\Theta_i \in H^{4i}(BO)$ *such that*

$$H^*(BO) \cong Z_p[\Theta_1, \Theta_2, \cdots]$$

and

$$H^*(BO\langle 4k\rangle) = \frac{H^*(BO)}{Z_p[\Theta_i \mid \sigma_p(2i-1) < 2k-1]} \otimes \bigotimes_{t\ \mathrm{odd}>0}^{p-2} F_{4k-3-2t},$$

the isomorphisms being as Hopf algebras over Z_p. *Moreover, the first factor is the image of the map induced by the projection* $BO\langle 4k\rangle \to BO$.

The following two lemmas from [4] will be needed in the proof.

LEMMA 1. *Given a diagram of the form*

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$$\begin{array}{ccc} K(Z, n-1) & \xrightarrow{=} & K(Z, n-1) \\ i \downarrow & & \downarrow \\ E & \rightarrow & PK(Z, n) \\ q \downarrow & & \downarrow \\ B & \xrightarrow{f} & K(Z, n) \end{array}$$

where E and B are homotopy associative and commutative h-spaces, q and f maps of h-spaces, and E the bundle induced by f from the path space fibration over K(Z, n) satisfying the following conditions

(i) $H^*(B)$ *is a free commutative* Z_p *algebra of finite type,*

(ii) E *is n-connected,*

(iii) *The* $\mathcal{A}_p$*-Hopf subalgebra* $\ker f^* \subset H^*(K(Z, n), Z_p)$ *is generated over* $\mathcal{A}_p$ *by the single element* $\beta\mathcal{P}^1\sigma_n$.

Then (1) $H^*(E)$ *is isomorphic as a tensor product of Hopf algebras to* $H^*(B)//\mathrm{Im}\, f^* \otimes F_{n-1}$,

(2) $\mathrm{Im}\, q^* = H^*(B)//\mathrm{Im}\, f^*$,

(3) $\mathrm{Im}\, i^* = F_{n-1}$.

For proof, see [4, Theorem 2.6] and Equation (4.13).

To prove Theorem 1, we will apply Lemma 1 inductively to the case $B = BO\langle 4k\rangle$, and $f: BO\langle 4k\rangle \rightarrow K(Z, 4k)$, where f induces an isomorphism from $H^{4k}(K(Z, 4k))$ to $H^{4k}(BO\langle 4k\rangle)$. But first we need a second result of [4].

LEMMA 2. *Consider the following diagram for each k*:

$$\begin{array}{ccc} K(Z, 2k-3) & \longrightarrow & K(Z, 2k-3) \\ i_k \downarrow & & \downarrow \\ BU\langle 2k\rangle & \longrightarrow & PK(Z, 2k-2) \\ q_k \downarrow & & \downarrow \\ BU\langle 2k-2\rangle & \xrightarrow[g_{k-1}]{} & K(Z, 2k-2) \end{array}$$

Then (1) $H^*(BU\langle 2k\rangle)$ *is isomorphic as a tensor product of Hopf algebras to*

$$\frac{H^*(BU)}{Z_p[\Theta_i' \mid \sigma_p(i-1) < k-1]} \otimes \bigotimes_{t=0}^{p-2} F_{2k-3-2t},$$

where $\Theta_i' \in H^{2i}(BU)$ *are chosen generators of*

$$H^*(BU) \simeq Z_p[\Theta_1', \Theta_2', \cdots].$$

$$(2)\qquad \operatorname{Im} q_k^* = \frac{H^*(BU)}{Z_p[\Theta_i' \mid \sigma_p(i-1) < k-1]} \otimes \bigotimes_{t=1}^{p-2} F_{2k-3-2t},$$

$$\operatorname{Im} i^* = F_{2k-3}.$$

$$(3)\qquad \operatorname{Im} g_k^* = Z_p[\Theta_i' \mid \sigma_p(i-1) = k-1] \otimes F_{2k-2p+1},$$

$$\operatorname{Ker} g_k^* = F_{2k}.$$

For the proof see [4, Theorem 4.1].

We now restate Theorem 1 with the complete induction hypothesis.

THEOREM 1′. *Consider the following diagram*

$$\begin{array}{ccccc}
K(Z, 4k-5) & \xrightarrow{=} & K(Z, 4k-5) & \xrightarrow{=} & K(Z, 4k-5) \\
\iota_k \downarrow & & \iota_k \downarrow & & \downarrow \\
BO\langle 4k\rangle & \xrightarrow[\bar\rho_{k-1}]{} & BU\langle 4k-2\rangle & \longrightarrow & PK(Z, 4k-4) \\
p_k \downarrow & & q_k \downarrow & & \downarrow \\
BO\langle 4k-4\rangle & \xrightarrow[\rho_{k-1}]{} & BU\langle 4k-4\rangle & \xrightarrow[g_{k-1}]{} & K(Z, 4k-4)
\end{array}$$

where ρ, $\bar\rho$ are the maps induced by the complexification. Let $f_k = g_k\rho_k$. Now $\bar\rho_0: BO \to BU$ and we define classes $\Theta_i \in H^{4i}(BO)$ by writing $\Theta_i = \rho_0^*(\Theta_{2i}')$, where the Θ_{2i}' are the classes of Lemma 2. Then

(1)$_k$ *$H^*(BO\langle 4k\rangle)$ is isomorphic as a tensor product of Hopf algebras to*

$$\frac{H^*(BO)}{[\Theta_i \mid \sigma_p(2i-1) < 2k-1]} \otimes \bigotimes_{t\ \mathrm{odd}>0}^{p-2} F_{4k-3-2t},$$

$$\operatorname{Im} p_k^* = \frac{H^*(BO)}{[\Theta_i \mid \sigma_p(2i-1) < 2k-1]} \otimes \bigotimes_{t\ \mathrm{odd}>1}^{p-2} F_{4k-3-2t},$$

$$\operatorname{Im} \iota_k^* = F_{4k-5}.$$

(2)$_k$ $$\operatorname{Ker} \rho_k^* = Z_p[\Theta_i' \mid i\ \mathrm{odd}, \sigma_p(i-1) \geqq 2k-1] \otimes \bigotimes_{t\ \mathrm{even}\geq 0}^{p-2} F_{4k-3-2t},$$

ρ_k^* is surjective.

(3)$_k$ $$f_k^*: H^{4k}(K(Z, 4k)) \to H^{4k}(BO\langle 4k\rangle)$$

is an isomorphism, and

$$\operatorname{Ker} f_k^* = F_{4k}, \qquad \operatorname{Im} f_k^* = Z_p[\Theta_i \mid \sigma_p(2i-1) = 2k-1] \otimes F_{4k-2p+1}.$$

Note that ρ_k^* restricted to the image of g_k^* is a monomorphism.

PROOF.

Step 1. *The inductive step*. We assume $(1)_j$, $(2)_j$, $(3)_j$ for $j \leq k$. We need then to prove $(1)_{k+1}$, $(2)_{k+1}$ and $(3)_{k+1}$. Now $(1)_{k+1}$ follows from $(3)_k$ and Lemma 1. To prove $(2)_{k+1}$, consider the diagram

$$\begin{array}{ccc}
BO\langle 4(k+1)\rangle & \xrightarrow{\rho_{k+1}} & BU\langle 4(k+1)\rangle \\
 & \searrow^{\bar{\rho}_k} & \downarrow \bar{q}_{k+1} \\
p_k \downarrow & & BU\langle 4k+2\rangle \\
 & & \downarrow q_{k+1} \\
BO\langle 4k\rangle & \xrightarrow[\rho_k]{} & BU\langle 4k\rangle
\end{array}$$

By the inductive hypothesis on ρ_k^*, it suffices to consider $\rho_{k+1}| F_{4k+1} \otimes F_{4k-1} \subset H^*(BU\langle 4(k+1)\rangle)$. But $\bar{\rho}_k = \bar{q}_{k+1}\rho_{k+1}$, and $i_{k+1}^* = \iota_{k-1}^* \bar{\rho}^*$, hence $\bar{\rho}_k^*(\beta \mathcal{P}^1 \sigma_{4k-1}) = \beta \mathcal{P}^1 \sigma_{4k-1} \in F_{4k-1} \subset H^*(BO\langle 4(k+1)\rangle)$. $(3)_{k+1}$. In $(2)_{k+1}$ we have shown that ρ_{k+1}^* is monomorphic on the image of g_{k+1}^*. Hence $\ker f_{k+1}^* = \ker g_{k+1}^* = F_{4k+4}$, and $\operatorname{Im} f_{k+1}^* = \rho^* \operatorname{Im} g_{k+1}^* = Z_p[\Theta_i | \sigma_p(2i-1) = 2k+1] \otimes F_{4k-2p+5}$. That f_{k+1}^* is an isomorphism in dimension $4k+4$ follows from the facts that ρ_{k+1}^* is surjective, g_{k+1}^* is an isomorphism, and $H^{4k+4}(BO\langle 4k+4\rangle) = Z_p$.

Step 2. *Low dimensions*. We have the diagram:

$$\begin{array}{ccccc}
 & K(Z,3) & \rightarrow & K(Z,3) & \rightarrow & K(Z,3) \\
 & \iota_2 \downarrow & & \downarrow & & \downarrow \\
 & BO\langle 8\rangle & \xrightarrow[\bar{\rho}_1]{} & BU\langle 6\rangle & \rightarrow & PK(Z,3) \\
 & p_2 \downarrow & & \downarrow & & \downarrow \\
BO \cong & BO\langle 4\rangle & \xrightarrow[\rho_1]{} & BU\langle 4\rangle & \xrightarrow[g_1]{} & K(Z,4)
\end{array}$$

Then $f_1^* = \rho_1^* g_1^*(\sigma_4) = \Theta_1 \in H^4(BO)$. Now ρ_1^* is surjective, and $\operatorname{Ker} \rho_1^* = Z_p[\Theta_i | i \text{ odd}]$. Also $\operatorname{Im} g_1^* = Z_p[\Theta_i' | \sigma_p(i-1) = 1]$. But $\sigma_p(i-1) = 1$ implies i is even, so ρ_1^* is a monomorphism on the image of g_1^*. Hence $\operatorname{Ker} g_1^* = \operatorname{Ker} f_1^* = F_4$, and $\operatorname{Im} f_1^* = Z_p[\Theta_i | \sigma_p(2i-1) = 1]$. Applying Lemma 1, we obtain $(1)_2$. The proof of $(2)_2$ and $(3)_2$ follows exactly as in the inductive step. To complete the proof of Theorem 1, note that the projection $p: BO\langle 4k\rangle \rightarrow BO$ is the composition $p_k p_{k-1} \cdots p_2$. Hence

$$p^* H^*(BO) \simeq \frac{H^*(BO)}{Z_p[\Theta_i | \sigma_p(2i-1) < 2k-1]}.$$

As an application, the following corollary is proven.

COROLLARY 1. *Let γ denote the universal vector bundle over BO, $p^*(\gamma)$ the induced bundle over $BO\langle 4k\rangle$, and $MO\langle 4k\rangle$ the Thom space of $p^*(\gamma)$. If $p>2k$, $H^*(MO\langle 4k\rangle)$ is a free module over the subalgebra $'\mathcal{A}_p$ of $\mathcal{A}_p$ generated by the reduced pth powers.*

PROOF. Consider the map $\alpha: '\mathcal{A}_p \rightarrow H^*(MO\langle 4k\rangle)$, $\alpha(1)=U$, the Thom class. Then by results of Milnor and Moore [3, Theorem 4.4 and Proposition 2.6], it is sufficient to show that α restricted to the primitive elements of $'\mathcal{A}_p$ is a monomorphism. Hence we need to know how the Steenrod algebra acts on $H^*(MO\langle 4k\rangle)$. First consider $H^*(BO)$. Here $\Theta_i \equiv P_i$ modulo decomposables, P_i being the ith Pontryagin class, since both the Θ_i's and the P_i's form a system of generators for the polynomial algebra $H^*(BO)$. Milnor [1] has shown that in $H^*(MO)$, $\mathcal{P}^i U \equiv aP_{(p-1)i/2}$ modulo decomposables, $0\neq a \in Z_p$, and Yokota [6] the formula

$$\mathcal{P}^i P_j = \binom{2j-1}{i} P_{(p-1)i/2+j}.$$

According to Milnor [2] the only primitive elements in $'\mathcal{A}_p$ are the elements λ_j, $j>0$, where $\lambda_1 = \mathcal{P}^1$ and $\lambda_{i+1} = [\mathcal{P}^{p^i}, \lambda_i]$. Putting everything together, one obtains $\lambda_j U = P_{(p^j-1)/2} = \Theta_{(p^j-1)/2}$ modulo decomposables. Hence we need only show that $p^*(\Theta_{(p^i-1)i/2}) \in H^*(BO\langle 4k\rangle)$ is nonzero modulo decomposables. By Theorem 1, this is true exactly when $\sigma_p(2(\frac{1}{2}(p^j-1))-1) \geqq 2k-1$. But $\sigma_p(2(\frac{1}{2}(p^j-1))-1) = \sigma_p(p^j-2)$ $= jp-2$. So we need $jp-2 \geqq 2k-1$ for all j, or $p-2 \geqq 2k-1$, or $p>2k$.

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