ON THE PL NONINVARIANCE OF THE SPAN OF A SMOOTH MANIFOLD

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1. Introduction. The purpose of this article is to construct examples illustrating a kind of pathology which arises because of the difference between the notions of vector bundle equivalence and PL microbundle equivalence. Our main result can be stated as follows.

THEOREM. *For every integer* $k \geqq 2$, *there is a closed* $(4k-2)$-*connected*, $(8k+1)$-*dimensional* PL *manifold* V *admitting two distinct smoothness structures* V_1 *and* V_2 *such that*

(a) span $(V_1)=$ span (V_2),

(b) *if* W *is any closed smooth* π-*manifold of dimension* $\geqq 1$, *then* span $(V_1\times W)\neq$ span $(V_2\times W)$.

The proof of this theorem is given in §2 below. In §3, we consider some related examples.

The material in this note is based on a portion of the author's doctoral thesis [8]. I would like to take this opportunity to express my gratitude to Professor M. Kervaire for his guidance and interest.

2. **Proof of the theorem.** Before beginning the proof proper, we recall some concepts and notation.

Let X be a finite CW-complex. For the purposes of this paper, X may be restricted to be simply-connected, so, for simplicity, we make this assumption on X. The groups $k_0(X)$, $k_{PL}(X)$ are as in Milnor [6], [7]. Elements of $k_0(X)$ (resp. $k_{PL}(X)$) will be referred to as 0-bundles (resp. PL-bundles). We shall make use of Wall's theory of thickenings, [14]. Recall that $T_0^m(X)$, the set of smooth m-thickenings of X, is the set of equivalence classes of pairs (M^m, ϕ), M a smooth, compact, oriented manifold with simply-connected boundary bM, $\phi: X\to M$ a homotopy equivalence (automatically simple); (M^m, ϕ) and (N^m, ψ) being regarded as equivalent if there exists an (oriented) diffeomorphism $h: M\to N$ such that $h\cdot\phi\simeq\psi$. There is a natural map $t(X): T_0^m(X)\to k_0(X)$, defined by sending the class of a pair (M^m, ϕ) into the pullback, under ϕ, of the stable tangent bundle of M. Moreover, there is an obvious suspension map $S: T_0^m(X) \to T_0^{m+1}(X)$, and $t(X)$ is compatible with S. Quite analogously, we can define $T_{PL}^m(X)$, the set of PL m-thickenings of X; $T_{PL}^m(X)$ enjoys properties similar to those of $T_0^m(X)$. Observe that there is a map $T_0^m(X)\to T_{PL}^m(X)$, deriving from Whitehead's theory of C^1 triangula-

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tions. Clearly, $t(X)$ "carries" this map into Milnor's map $k_0(X) \to k_{PL}(X)$. We note the following two key facts, both proved in Wall [14].

(1.1) The sets $T_0^m(X)$, $T_{PL}^m(X)$ stabilize. Precisely, the maps $T_0^m(X) \to^S T_0^{m+1}(X)$, $T_{PL}^m(X) \to^S T_{PL}^{m+1}(X)$ are bijective, for $m \geqq 2 \cdot \dim (X)+1$. Thus, we have stable sets $T_0(X)$, $T_{PL}(X)$.

(1.2) The natural maps $T_0(X) \to^{t(X)} k_0(X)$, $T_{PL}(X) \to^{t(X)} k_{PL}(X)$ are bijective.

We now prove the theorem. Let $X = S^{4k-1} \cup_q e^{4k}$, with q a prime dividing $(2^{2k-1}-1)\cdot \mathrm{num}(B_k/k)$. According to Milnor [7], the map $k_0(X) \to k_{PL}(X)$ is the zero map, although $k_0(X) \approx Z_q$. Let then $\xi \in k_0(X)$ be nonzero and let ϵ denote the trivial 0-bundle. By (1.1) and (1.2), we can find smooth thickenings (M_ξ^{8k+1}, ϕ), $(M_\epsilon^{8k+1}, \psi)$ of X such that $t(X)(M_\xi, \phi) = \xi$, $t(X)(M_\epsilon, \psi) = \epsilon$. Moreover, since ξ and ϵ are PL-equivalent, M_ξ and M_ϵ are PL-homeomorphic; let M be the underlying PL manifold of M_ξ and M_ϵ. We define V to be the PL double of M. Then V admits the smoothness structures V_1, V_2 obtained by smoothly doubling M_ξ, M_ϵ. Observe that V_2 is a π-manifold, whereas V_1 is not.

We verify (a) and (b) of the theorem. We begin by proving that span $(V_2)=1$. Indeed, since V_2 is a π-manifold, a theorem due to Bredon-Kosinski [2] and Thomas [13] implies that span (V_2) is either equal to dim $(V_2) = 8k+1$ or to span $(S^{8k+1}) = 1$. But according to a criterion of Bredon-Kosinski (loc. cit.), span $(V_2) =$ span (S^{8k+1}) iff the reduced semicharacteristic $\hat{\chi}(V_2) = 1 - \chi^*(V_2)$ is zero. We therefore calculate

$$\chi^*(V_2) = \sum_{i=0}^{4k} b_i(V_2),$$

$b_i(V_2)$ denoting the ith mod 2 Betti number. Since V_2 is connected and simply-connected (van Kampen), we have $b_0(V_2)=1$, $b_1(V_2)=0$. Using Poincaré-Lefschetz duality and the mod 2 Mayer-Vietoris sequence of the triad $(V_2, M_\epsilon, \overline{M}_\epsilon)$ (V_2 being the union of two copies M_ϵ, $\overline{M}_\epsilon$ of the manifold M_ϵ intersecting along their common boundary) we deduce easily that V_2 is a Z_2-homology sphere, so that in particular,

$$b_2(V_2) = \cdots = b_{4k}(V_2) = 0. \qquad \text{Thus, } \chi^*(V_2) = 1, \quad \hat{\chi}(V_2) = 0$$

and our contention is proved.

Since V_1 and V_2 are homeomorphic, we can now appeal to a theorem of Haefliger-Hirsch [3] to deduce that span $(V_1) =$ span $(V_2) = 1$.

(Recall that the theorem of Haefliger-Hirsch states that if W_1^n and W_2^n are homeomorphic smooth closed manifolds, and $k \leqq \frac{1}{2}(n-1)$, then span $(W_1) \geqq k$ iff span $(W_2) \geqq k$.) Thus, (a) is verified. To prove (b), note that $V_1 \times W$ is still not a π-manifold (so that span $(V_1 \times W) <$ dim $(V_1 \times W)$) while $V_2 \times W$ is actually parallelizable (i.e. span $(V_2 \times W) =$ dim $(V_2 \times W)$). This latter statement is true because V_2 has zero Euler characteristic (cf. Staples [**10**]). This completes the proof.

REMARK. It was not necessary to choose the thickenings M_ξ, M_ϵ to be of dimension $8k+1$; any odd dimension $m \geqq 8k+1$ would have sufficed. For the appeal to the Haefliger-Hirsch theorem would still be justified since span $(S^m) < \frac{1}{2}(m-1)$ ([cf. Adams [**1**]).

3. Further examples and remarks. We consider again the complex $X = S^{4k-1} \cup_q e^{4k}$ but we remove the Milnor divisibility restriction on q and impose the restriction $q > 2k+1$.

LEMMA. *Any $\xi \in k_0(X)$ is fiber homotopy trivial.*

PROOF. Let $k_H(X)$ be the group of stable spherical fibrations over X. Recall that if BH is the classifying space for such fibrations, then $k_H(X) \approx [X, BH]$. The Eckmann-Hilton Universal Coefficient Theorem (cf. [**4**]) gives an exact sequence

$$0 \to \operatorname{Ext}(\boldsymbol{Z}_q, \pi_{4k}(BH)) \to [X, BH] \to \operatorname{Hom}(\boldsymbol{Z}_q, \pi_{4k-1}(BH)) \to 0.$$

We now identify $\pi_{4k}(BH)$, $\pi_{4k-1}(BH)$ with the stable stems π_{4k-1}^S, π_{4k-2}^S and recall that the latter have no q-torsion for $q > 2k+1$. Thus $[X, BH] = 0$ and the lemma is proved.

For $\xi \in k_0(X)$, we form M_ξ^m, M_ϵ^m and V_1^m, V_2^m as before (m being a "large" odd integer). We would like to say that V_1 and V_2 are homotopy equivalent since ξ and ϵ are fiber homotopy equivalent but this is not immediately apparent. It is, of course, sufficient to show that (M_ξ, bM_ξ) and $(M_\epsilon, bM_\epsilon)$ are homotopy equivalent as pairs. We indicate how this can be proved. Just as in the smooth and PL categories, there is, for any finite CW-complex X, a set $T_H^m(X)$, the set of "homotopy" m-thickenings of X. This is defined by replacing the words "manifold" and "manifold-with-boundary" by "P-space" and "P-pair"; cf. Spivak [**9**], Levitt [**5**]. The analog of diffeomorphism (or PL-homeomorphism) of manifolds is equivalence of P-spaces or P-pairs, i.e. ordinary homotopy equivalence of spaces or pairs, and the analog of the stable tangent bundle (or microbundle) of a manifold is the Spivak normal fibration (or rather its inverse) of a P-space or P-pair. Hence, as in the other categories, we have natural maps $t(X): T_H^m(X) \to k_H(X)$ which are compatible with suspension. Moreover, there are also analogs of (1.1) and (1.2), i.e. $T_H^m(X)$ stabilizes for m large (compared to dim (X)), and the induced map $t(X): T_H(X) \to k_H(X)$ on stable objects is bijective. These two results are due to Levitt [5].

Returning to the situation considered above, we see that for m large, the manifold pairs (M_ξ^m, bM_ξ^m) and $(M_\epsilon^m, bM_\epsilon^m)$ are equivalent as P-pairs, which establishes our claim.

We assert that there is an analog of the theorem of §1 for our present manifolds V_1^m, V_2^m (m odd). Part (b) of the theorem goes exactly as before. To prove part (a), we evidently need a version of the Haefliger-Hirsch theorem for homotopy equivalent smooth manifolds. According to Sutherland (private communication), it follows from results of Hirsch and Wagoner that the Haefliger-Hirsch theorem is true for homotopy equivalent manifolds provided these manifolds are odd-dimensional and 2-connected (which is certainly true in our case). Alternatively, we can use the weaker result of Sutherland [12]. Observe that the hypothesis of Sutherland's Theorem 1.2c is trivially satisfied in our case (for appropriate choice of dim (V_1) $=$dim (V_2)) since V_2 is a π-manifold.

Remarks. (1) If the prime q (satisfying $q > 2k+1$) is large enough, then the homotopy equivalent manifolds are not PL-homeomorphic. In fact, for every integer $k \geqq 1$, there exists a positive integer c_k such that whenever W_1 and W_2 are PL-homeomorphic smooth manifolds, $p_k(W_1) - p_k(W_2)$, the difference of the integral Pontryagin classes, has finite order $\leqq c_k$; cf. [8]. But clearly, the order of $p_k(V_1) - p_k(V_2)$ is not less than the order of $p_k(M_\xi) - p_k(M_\epsilon) = p_k(\xi) - p_k(\epsilon) = p_k(\xi)$, and $p_k(\xi)$ has order q by Milnor [7]. Thus, if $q > c_k$, V_1 and V_2 are combinatorially distinct.

(2) It is easy to see that, as a consequence of Sullivan's solution of the Hauptvermutung (cf. [11]), V_1 and V_2 are not homeomorphic. Thus, we have obtained new (and stronger) counterexamples to the Hurewicz conjecture; namely, we have nonhomeomorphic, simply-connected, homotopy equivalent smooth manifolds with identical rational Pontryagin classes.

Added in proof. Other, different sorts of counterexamples to the Hurewicz conjecture have been given by S. P. Novikov (*Homotopically equivalent smooth manifolds. I*, Amer. Math. Soc. Transl. (2) **48** (1965), 271–396) and D. Sullivan (Thesis, Princeton Univ., Princeton, N. J., 1965). In these examples, the manifolds are actually tangentially homotopy equivalent (so that even the integral Pontrjagin classes coincide); the invariant, which in each case distinguishes the manifolds combinatorially (and hence, by Sullivan's Hauptvermutung, topologically) is a certain *mod* 2 cohomology class which represents a "surgery obstruction."

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