DEFORMATION RETRACTS AND WEAK DEFORMATION RETRACTS OF NONCOMPACT MANIFOLDS[1]

GUDRUN KALMBACH

1. **Introduction.** This note gives a proof of the following result: Let M be either a smooth noncompact manifold with a countable base or a compact manifold with nonempty boundary. Let f be a nondegenerate function on M with a finite number of critical points $p_1, \cdots, p_r$ with indices $\lambda_1, \cdots, \lambda_r$ resp., where in the noncompact case $f(q_i) \to \infty$ for each sequence $q_i \in M$ without limit point or in the compact case $f(p) = a$ on the boundary points p for some fixed $a \in \boldsymbol{R}$. If E_i is the open descending λ_i-dimensional bowl, determined by p_i, [4], and $K = \bigcup_{i=1,2,\ldots} E_i$, then K is a deformation retract of M.

Functions with an infinite number of critical points on M do not have this property in general: there exists a simple example of a two-dimensional manifold M and a nondegenerate function f on M with an infinite number of critical points of index 1, a single critical point of index 0 and no critical point of index 2, such that K is not a deformation retract of M.

But the theorem above holds in such a case in the form that K is a "weak deformation retract," (see §4), of M.

2. **Notation and preliminary notes.** The notation defined in this paragraph will be used throughout this note.

$M = M^n$ is a smooth, noncompact manifold of dimension n with a countable base.

f is a nondegenerate function on M, [1], with critical points $p_1, p_2, p_3, \cdots$ and $f(p_1) < f(p_2) < f(p_3) < \cdots$ and with $f(q_k) \to \infty$ if the sequence $q_k \in M$ has no limit point in M.

λ_i is the index of f at p_i.

E_i is the descending λ_i-bowl associated with p_i, [4, p. 83].

D_M is the set of smooth functions on M.

$\Delta f(p)$ is the gradient of f at p.

Presented to the Society, January 26, 1968 under the title *Some remarks on noncompact manifolds*; received by the editors October 26, 1967.

[1] This is a part of the author's dissertation, *Über niederdimensionale CW-Komplexe in nicht-kompakten Mannigfaltigkeiten*, Göttingen, 1966, thesis adviser: Professor H. Grauert.

The writer thanks Professor S. S. Cairns for discussion and suggestions.

This report was prepared under grant NSF GP 5610.

For each noncritical point p of M, (that is, for each point p where $\Delta f(p)$ is not zero), there is an orthogonal trajectory to the level surfaces of f through p.

$\phi(p)$ denotes the (maximal) orthogonal trajectory through p to the level surfaces of f such that $\phi(p)$ contains as subsets all other orthogonal trajectories of f through p; we write also

ϕ for the set $\{\phi(p) \mid p \in M\}$. (If p is a critical point of f then we set $\phi(p) = \{p\}$.)

$$K = \bigcup_{i=1,2,\cdots} E_i, \quad M^a = \{p \in M \mid f(p) \leqq a\}, \quad \partial M^a = \{p \in M \mid f(p) = a\}.$$

It is possible to choose the Riemannian metric on M such that for each point p_i there exists a neighborhood U_i in M with coordinate functions x in U_i such that $f(x)$ is given in U_i by

$$f(x) = -\sum_{k=1}^{\lambda_i} x_k^2 + \sum_{k=\lambda_i+1}^{n} x_k^2 + f(p_i)$$

and the system of differential equations of ϕ in U_i is given by

$$dz_k(t)/dt = -2z_k \qquad \text{for } k = 1, \cdots, \lambda_i,$$
$$dz_k(t)/dt = 2z_k \qquad \text{for } k = \lambda_i + 1, \cdots, n.$$

Then ϕ is locally given by $\phi(t, x) = (e^{-2t}x_1, \cdots, e^{-2t}x_{\lambda_i}, e^{2t}x_{\lambda_i+1}, \cdots, e^{2t}x_n)$. Let $A \subset M$ and $B \subset M$ be closed sets in M with $A, B \neq \emptyset$ and $A \cap B = \emptyset$. A function $g \in D_M$ is called a *separation-function* of A and B in M if $g(p) = 0$ for $p \in A$ and $g(p) = 1$ for $p \in B$, if $0 \leqq g(p) \leqq 1$ in M and $\Delta g(p) \neq 0$ for all points $p \in M$ with $0 < g(p) < 1$.

REMARK 1. If $U \subset M$ is open and $h \in D_u$; if $a < b$ and $B = \{p \in U \mid h(p) \leqq a\} \neq \emptyset$, if $A = U - \{p \in U \mid h(p) < b\} \neq \emptyset$ and the gradient Δh of h is $\neq 0$ on $\{p \in U \mid a \leqq h(p) \leqq b\} = C$, then there exists a separation-function g of A and B in U such that $\Delta g(p) \neq 0$ in Int (C) and the level manifolds of g on C are the same as the level manifolds of h.

If one takes the function

$$G(y) = \frac{\int_y^b \exp[-1/(t-a) + 1/(t-b)]dt}{\int_a^b \exp[-1/(t-a) + 1/(t-b)]dt}$$

for real numbers y with $a \leqq y \leqq b$ and $G(y) = 0$ if $y \geqq b$, respectively $G(y) = 1$ if $y \leqq a$; if we take $g(p) = G(h(p))$ then all conditions of the remark are fulfilled.

$g_{a,b,h} = g$ always means the function g of this remark, it depends on the numbers a, b and the function h.

REMARK 2. Let $U \subset M$ be open and $h \in D_u$. Let

$$G = \{p \in U \mid h(p) = d\}$$

with $d \in \mathbf{R}$, let $\Delta h(p) \neq 0$ if $p \in G$. Then the set χ of maximal orthogonal trajectories $\chi(p)$ of h, $(p \in G)$, is defined in U and for suitable parameters of χ in $N = \{q \in \chi(p) \mid p \in G\}$ and suitable neighborhoods U_1 and U_2 of a point $p \in G$ and $\chi_t(p) \in N$ respectively the correspondence

$$\chi_t : \left\{ \begin{array}{c} U_1 \to U_2 \\ p \to \chi_t(p) \end{array} \right\}$$

is a diffeomorphism ([1, p. 13], t is fixed). Therefore, if $F \in D_G$ and $q = \chi_t(p)$, then $g(q) = g(\chi_t(p)) = F(p)$, $(p \in G)$, defines a function g in D_N and if $\Delta F(p) \neq 0$ for all $p \in G$ then $\Delta g(q) \neq 0$ for all $q \in N$.

REMARK 3. $\bar{E}_i \subset K$.

PROOF. Let q_i be a sequence of points in E_i which has as limit point p some point of M not in E_i, then it may happen that $p = p_k$, is a critical point; in which case $p \in K$. If not, then only a finite number of points q_i can be contained in a maximal orthogonal trajectory of f, that means that there exists an infinite subsequence $q_1', q_2', \cdots$ of q_i with $\phi(q_m') \cap \phi(q_e') = \emptyset$ if $m \neq e$. These orthogonal trajectories converge in a neighborhood of p to the orthogonal trajectory $\phi(p)$ and in U_i (a neighborhood of p_i) to an orthogonal trajectory $g \subset E_i$, [2]. It follows that $\phi(p) \cap g = \phi$ since $p \notin E_i$, so $\phi(p)$ has a critical point p_k as a boundary point in $M - M^{f(p)}$, so $p \in \phi(p) \subset E_k \subset K$.

3. Special vector fields on subsets of M. Let $\epsilon_i > 0$ be a suitable constant,

$$V_i'' = \left\{ x \in U_i \mid h_i''(x) = \sum_{j=\lambda_i+1}^{n} x_j^2 < \epsilon_i \right\}$$

and

$$V_i' = \{p \in \phi(x) \cap M^{f(x)} \mid x \in V_i''\}.$$

PROPOSITION 1. *Let $\lambda_i < n$. There exists a function $h_i \in D_{V_i'}$ such that $\Delta h_i(p) = 0$ if $p \in E_i$ and $\Delta h_i(p) \neq 0$ if $p \in V_i' - E_i$.*

PROOF. Take $a' < b' < f(p_i)$ with $(\partial M^{a'}) \cap V_i' \subset U_i$ and $g(p) = g_{a',b',f}(p)$ if $p \in V_i'$. If $x \in U_i$ and $x' = \phi(x) \cap \partial M^{a'}$, then

$$x' = (e^{-2t}x_1, \cdots, e^{-2t}x_{\lambda_i}, e^{2t}x_{\lambda_i+1}, \cdots, e^{2t}x_n)$$

for some $t \in \mathbf{R}$ (see §2). The function $h^+(p) = h_i''(\phi_t(p))$ with $\phi_t(p)$

$=\phi(p)\cap\partial M^{a'}\cap V_i'$ is defined in $V_i'-\{p_i\}$ according to Remark 2. It follows that $0\leqq h^+(x)=e^{4t}h_i''(x)\leqq h_i''(x)$ if $t\leqq 0$ and $x\in U_i$. Define $h_i(p)=(1-g(p))h_i''(p)+g(p)h^+(p)$. If $x\in U_i\cap(M^{b'}+\text{Int}(M^{a'}))$ and $x\notin E_i$ then

$$\Delta f(h_i(x)) = (1-g(x))\Delta f(h_i''(x)) + (-h_i''(x)+h^+(x))\Delta f(g(x)) > 0$$

so $\Delta h_i(x)\neq 0$. If $x\in E_i\cap U_i$ then $\Delta h_i(x)=0$. It follows from the preceding two sentences and from the definition of h_i that $\Delta h_i(p)=0$ if $p\in E_i$, otherwise $\Delta h_i(p)\neq 0$ in V_i' and that $\Delta h_i(p)$ is tangent to the level surface ∂M^c if $p\in\partial M^c\cap V_i'-U_i$.

Some notation. Let $0<2\epsilon_i'<\epsilon_i$ and $g_i(p)=g_{\epsilon_i',2\epsilon_i',h_i}(p)$ if $p\in V_i'$ (Remark 1); let $W_i'=\{p\in V_i'\mid g_i(p)=0\}$ and $W_i=\{p\in V_i'\mid g_i(p)=1\}$. We take an arbitrary noncritical value c of f and the set $K^c=\bigcup_{i:p_i\in M^c}E_i$ and we will show that K^c is a deformation retract of M^c. We take first exactly the critical points $p_{i_1}, p_{i_2},\cdots,p_{i_s}$ with the properties $p_{i_v}\in M^c$ and $\lambda_{i_v}<n$. Let $V_e=\bigcup_{v=e}^s V_{i_v}'$, $1\leqq e\leqq s$; let $U_e'=\bigcup_{v=e}^s W_{i_v}$, $U_e^+=\bigcup_{v=e}^s(V_{i_v}'-W_{i_v}')$, $B^e=\bigcup_{v=e}^s E_{i_v}$ and $U^c=\bigcup_{v=1}^s U_{i_v}$.

PROPOSITION 2. *There exists a separation function g_k' of $V_k-U_k^+$ and U_k' in V_k.*

PROOF. If $k=s$ then such a function exists, namely $g_s'=g_{i_s}$. For some $k\leqq s$ assume that the function g_k' exists, then

$$g_{k-1}'(p) = 1-(1-g_k'(p))(1-g_{i_{k-1}}(p))$$

has the desired properties for $k-1$ instead of k.

PROPOSITION 3. *If there exists a number k with $1\leqq k\leqq s$ and a vector field X_k on U_k' such that $X_k(p)=0$ if $p\in B^k$ and $X_k(p)\neq 0$ if $p\in U_k'-B^k$ and $X_k(p)$ is tangent to the level surface ∂M^a if $p\in\partial M^a\cap(U_k'-B^k\cup U^c)$ with $a\in\mathbf{R}$, then there exists a vector field X_{k-1} on U_{k-1}' with the properties of X_k mentioned above (where k is replaced by $k-1$).*

PROOF. We take $X_{k-1}'(p)=g_k'(p)X_k(p)+(1-g_k'(p))\Delta h_{i_{k-1}}(p)$ in $V_{k-1}'-E_{i_{k-1}}$. Without restriction, $X_k(f(p))=0$ if $p\in V_{k-1}'\cap V_k$. We have

$$X_{k-1}'(f(p)) = (1-g_k'(p))\Delta h_{i_{k-1}}(f(p)) \geqq 0$$
$$> 0$$

if $(1-g_k'(p))\neq 0$ and $\Delta h_{i_{k-1}}(f(p))\neq 0$. This is the case if a', h'' are taken as in Proposition 1 if $h_{i_{k-1}}''(p)>0$ and $f(p)>a'$ i.e., if $p\in U_{i_{k-1}}-E_{i_{k-1}}$ with $f(p)>a'$ and $(1-g_k'(p))\neq 0$. Therefore $X_{k-1}'(p)\neq 0$ in $V_{k-1}'-B^{k-1}$. Take $g'(p)=1-g_{0,\epsilon_{k-1}'',h_{i_{k-1}}}(p)$ with $0<\epsilon_{k-1}''<\epsilon_{i_{k-1}}'$ and

$X_{k-1}(p)=g'(p)X'_{k-1}(p)$. Then X_{k-1} is a smooth vector field on $V'_{k-1} \cap U'_{k-1}$ since $g' \cdot g'_k \in D_{V'_{k-1}}$. The vector field X_{k-1} has the desired properties.

PROPOSITION 4. *Let $V^c = U'_1 \cup K^c$. Then K^c is a deformation retract of V^c.*

PROOF. Proposition 1 gives a vector field $X_s(p)=\Delta h_{i_s}(p)$ with the properties of Proposition 3 and $k=s$. So Proposition 3 gives a vector field X_1 on U'_1 with the properties of X_k (with $k=1$). V^c is a neighborhood of K^c in M^c. To each point $p \in V^c - K^c$ there exists a maximal trajectory $\chi(p)$ of X_1 through p and $\chi(p)$ has exactly one limit point $p' \in K^c$. Let d_p be the length of the segment of $\chi(p)$ between p and p'. If $p \in K^c$ we define $d_p=0$ and $\chi(p)=\{p\}$. Then d_p is continuous. The correspondence $p \to q \in \chi(p)$ with $d_q=(1-t)d_p$ and $0 \leqq t \leqq 1$ gives a deformation retract of V^c on K^c.

4. **Proof of the theorem.** Let K be defined as in §1 and K^c as in §3. Then we call K a weak deformation retract of M, if for each (noncritical) value $c \in \boldsymbol{R}$ the set K^c is a deformation retract of M^c.

THEOREM. *K is a weak deformation retract of M.*

PROOF. Take ϵ'_i and $i_1, \cdots, i_s$ as in §3 and $0<2a_i<\epsilon'_i$. Let $A = \bigcup_{i:\lambda_i=n;\, p_i \in M^c} E_i$ and $f_i(p)=g_{a_i,2a_i,h_i}(p)$ if $p \in V'_i - A$ and let $g^+(p) = 1-\prod_{\nu=1}^{s}(1-f_{i_\nu}(p))$. If $p \in A$ or $p \in M^c - V^c$ then we define $g^+(p)=1$ or $g^+(p)=0$ respectively. Then $g^+ \in D_{M^c}$ (Remark 3). The sets $B=\{p \in M^c \mid g^+(p)=0\}$ and $C=\{p \in M^c \mid g^+(p)=1\}$ are disjoint and closed in M^c, so g^+ is a separation function of B and C in M^c. Let $X_1(p)$ be as in Proposition 4 if $p \in V^c - A$ and $X_1(p)=0$ if $p \in A$; then the vector field $X^*(p)=g^+(p)X_1(p)+(1-g^+(p))\Delta f(p)$ is smooth on M^c. We have $X^*(p) \neq 0$ if $g^+(p) \neq 1$ since $X_1(f(p)) \geqq 0$.

The deformation retract of M^c on K^c is given in the following way: let d_p be the length of the segment of the maximal trajectory $\psi(p)$ of X^* between p and K^c if $p \in M^c - K^c$ and let $d_p=0$ and $\psi(p)=\{p\}$ if $p \in K^c$. Then the correspondence $p \to q \in \psi(p)$ with $d_q=(1-t)d_p$ and $0 \leqq t \leqq 1$ gives the desired deformation retract.

COROLLARY. *If f has only a finite number of critical points on M, then K is a deformation retract of M. This is also true if M is a manifold with boundary.*

The only remark to make is that if we take a noncritical value $c \in \boldsymbol{R}$ such that $\Delta f(p) \neq 0$ for all $p \in M - M^c$ then $X^* = \Delta f$ in a neighborhood of ∂M^c in M^c, so with the additional definition $X^*(p)=\Delta f(p)$

if $p \in M - M^c$ we can construct the deformation retract as in the proof of the theorem.

References

1. J. Milnor, *Morse theory*, Ann. of Math. Studies No. 51, Princeton Univ. Press, Princeton, N. J., 1963.

2. P. Montel, *Sur l'intégrale supérieure et l'intégrale inférieure d'une équation differentielle*, Bull. Sci. Math. (2) 50 (1926), 205–217.

3. M. Morse, *The existence of polar nondegenerate functions on differentiable manifolds*, Ann. of Math. 71 (1960), 352–383.

4. ———, *Bowls of a nondegenerate function on a compact differentiable manifold*, Differential and Combinatorial Topology, Princeton Univ. Press, Princeton, N. J., 1965, 81–103.

University of Illinois