A GENERALIZATION OF QUASI-FROBENIUS RINGS

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As a generalization of quasi-Frobenius rings, G. Azumaya [2] has investigated a ring with the property that every faithful left module is a generator[1] in the category of left modules, and he has proved that such a ring is left self-injective and a direct sum of indecomposable left ideals having minimal left ideals.

In his proof, however, the existence of the faithful injective module plays an important role, while the injective module is not necessarily finitely generated,[2] even if the ring is a left Artinian ring. Therefore, there naturally arises a problem: Is an Artinian ring quasi-Frobenius if every finitely generated faithful module is a generator?

The purpose of this paper is to give an affirmative answer to this problem as a direct consequence from a more general result which is rather similar to that of Azumaya stated above and related to perfect rings introduced by H. Bass [3].

Throughout this paper we shall assume that the ring R has identity element 1 and all modules over it are unital. For a subset A of R we shall denote

$$r(A) = \{x \mid Ax = 0, x \in R\}, \qquad l(A) = \{x \mid xA = 0, x \in R\}.$$

1. **Preliminaries.** Let R be a ring. Let M_1 and M_2 be left R-modules and T_1, T_2 submodules of them respectively. Assume that there exists a left R-isomorphism θ: $T_1 \to T_2$ (onto). Let us denote by L the factor module of $M_1 \oplus M_2$ by the submodule consisting of all elements of form $\{m, -\theta(m)\}$ for $m \in T_1$. Then there are the canonical injections α and β of M_1 and M_2 into L respectively. In [12] we called L the interlacing module of M_1 and M_2 by using θ as the lacing isomorphism and denote it by $\mathrm{Int}_\theta(M_1, M_2)$. In this paper, especially θ will be said to be maximal,[3] if there is no isomorphism which is an extension of θ.

In [3] H. Bass proved that for a right perfect ring R the Jacobson radical N of R is right T-nilpotent and the residue class ring $\overline{R} = R/N$ is semisimple Artinian. Let e be a primitive idempotent of R. The R-endomorphism ring of Re is inverse-isomorphic to eRe, and eRe is

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[1] In his paper, G. Azumaya calls a generator a completely faithful module.

[2] Cf. P. M. Cohn [5] and A. Rosenberg and D. Zelinsky [11].

[3] The definition of "maximality" is different from that of [12].

completely primary (i.e. nonregular elements form the unique maximal ideal), because eNe is nilideal. By the socle of a module we mean the sum of all simple submodules. It is known that every nonzero left R-module has nonzero socle if R is a right perfect ring.

One of the following properties of modules is retained under the category-isomorphism: (a) simple, (b) finitely generated, (c) projective, (d) injective, (e) faithful, (f) generator. As we are concerned only with the above properties, we shall assume throughout that R is isomorphic to its basic subring. Then, if $\{e_\kappa\}$, $\kappa=1, 2, \cdots, n$, are mutually orthogonal primitive idempotents of R such that $\sum_{\kappa=1}^{n} e_\kappa =1$, Re_κ is isomorphic to Re_λ if and only if $\kappa=\lambda$.

2. **Proof of the main theorem.** Assume that R is a right perfect ring. First we shall prove

PROPOSITION 2.1. *If every finitely generated faithful left R-module is a generator, then it follows that*

(1) *$e_\lambda r(N)$ is a simple right ideal of R.*

(2) *$r(N)e_\kappa$ is a simple left ideal of R.*

PROOF. Suppose $e_\lambda r(N)$ is nonzero right ideal. Consider the case (a): $e_\lambda r(N)e_\kappa \neq 0$ and $e_\lambda r(N)e_\mu \neq 0$ for $\kappa\neq\mu$. Take nonzero elements c_1 and c_2 such that $c_1 \in e_\lambda r(N)e_\kappa$ and $c_2\in e_\lambda r(N)e_\mu$. Denote by θ the left R-isomorphism: $Re_\lambda c_1 \rightarrow Re_\lambda c_2$, defined by $\theta(xe_\lambda c_1)=xe_\lambda c_2$ for $xe_\lambda c_1 \in Re_\lambda c_1$. Let us denote $\sum_{\rho\neq\kappa,\rho\neq\mu} \oplus Re_\rho \oplus \mathrm{Int}_\theta(Re_\kappa, Re_\mu)$ by L. Then L is a finitely generated, faithful left R-module, and hence by the assumption it follows that L is a generator. Thus the trace ideal of L must be R, and consequently there exist a family $\{x_i\}$ of elements of L and a family $\{\phi_i\}$ of left R-homomorphisms ϕ_i: $L\rightarrow R$ such that $\sum_{i=1}^{n} \phi_i(x_i) = 1$. Since L is a direct sum of Re_ρ's, $\rho\neq\kappa$, $\rho\neq\lambda$ and $\mathrm{Int}_\theta(Re_\kappa, Re_\mu)$, ϕ_i can be expressed as follows: $\phi_i=\sum_{\rho\neq\kappa,\rho\neq\mu} \phi_{\rho,i} + \phi_{(\kappa,\mu),i}$, where $\phi_{\rho,i}$, $\phi_{(\kappa,\mu),i} \in \mathrm{Hom}_R(L, R)$, $\phi_{\rho,i}(Re_{\rho'}) = 0$ for $\rho'\neq\rho, \kappa, \mu$, $\phi_{\rho,i}(\mathrm{Int}_\theta(Re_\kappa, Re_\mu))=0$, and $\phi_{(\kappa,\mu),i}(Re_\rho)=0$. On the other hand, to each $\phi_{\rho,i}$ and $\phi_{(\kappa,\mu),i}$ there exist an element $e_\rho a_{\rho,i}$ of R and a pair of elements $e_\kappa a_{\kappa,i}$ and $e_\mu a_{\mu,i}$ of R such that $\phi_{\rho,i}(x_\rho)=x_\rho e_\rho a_{\rho,i}$ for all $x_\rho \in Re_\rho$ and $\phi_{(\kappa,\mu),i}(x_\kappa\alpha+x_\mu\beta)=x_\kappa e_\kappa a_{\kappa,i}+x_\mu e_\mu a_{\mu,i}$ for all $x_\kappa\in Re_\kappa$ and for all $x_\mu\in Re_\mu$. Here we notice that $c_1e_\kappa a_{\kappa,i}=c_2e_\mu a_{\mu,i}$, for $c_1e_\kappa\alpha-c_2e_\mu\beta=0$ and $\phi_{(\kappa,\mu),i}(c_1e_\kappa\alpha-c_2e_\mu\beta)=0$. Then, putting $x_i=\sum_{\rho\neq\kappa,\rho\neq\mu} r_{\rho,i}e_\rho+r_{\kappa,i}e_\kappa\alpha +r_{\mu,i}e_\mu\beta$, for $r_{\rho,i}$, $r_{\kappa,i}$, $r_{\mu,i}\in R$, we have

$$\sum_i \phi_i(x_i) = \sum_{\rho\neq\kappa,\rho\neq\mu} \left(\sum_i r_{\rho,i}e_\rho a_{\rho,i}\right) + \sum_i r_{\kappa,i}e_\kappa a_{\kappa,i} + \sum_i r_{\mu,i}e_\mu a_{\mu,i} = 1.$$

It follows that

$$\sum_{\rho\neq\kappa,\rho\neq\mu}\sum_i e_\kappa r_{\rho,i}e_\rho a_{\rho,i}e_\kappa+\sum_i e_\kappa r_{\kappa,i}e_\kappa a_{\kappa,i}e_\kappa+\sum_i e_\kappa r_{\mu,i}e_\mu a_{\mu,i}e_\kappa=e_\kappa$$

and hence $\sum_i e_\kappa r_{\kappa,i}e_\kappa a_{\kappa,i}e_\kappa\equiv e_\kappa \bmod e_\kappa N^2e_\kappa$. Since $e_\kappa Re_\kappa$ is completely primary, we can assume $e_\kappa r_{\kappa,1}e_\kappa$ and $e_\kappa a_{\kappa,1}e_\kappa$ are two regular elements of $e_\kappa Re_\kappa$.

Now, by a similar argument we can assume that $e_\mu r_{\mu,j}e_\mu$ and $e_\mu a_{\mu,j}e_\mu$ are two regular elements of $e_\mu Re_\mu$. Therefore $c_1e_\kappa=c_2e_\mu a_{\mu,1}(e_\kappa a_{\kappa,1}e_\kappa)^{-1}$ and $c_1e_\kappa a_{\kappa,j}(e_\mu a_{\mu,j}e_\mu)^{-1}=c_2e_\mu$ and consequently $c_1e_\kappa=c_1e_\kappa a_{\kappa,j}(e_\mu a_{\mu,j}e_\mu)^{-1}\cdot e_\mu a_{\mu,1}(e_\kappa a_{\kappa,1}e_\kappa)^{-1}$. If we put $e_\kappa a_{\kappa,j}e_\mu(e_\mu a_{\mu,j}e_\mu)^{-1}e_\mu a_{\mu,1}e_\kappa(e_\kappa a_{\kappa,1}e_\kappa)^{-1}=n$, then $n\in e_\kappa N^2e_\kappa$ and $c_1e_\kappa(1-n)=0$. It follows that $c_1e_\kappa=0$, because $e_\kappa Ne_\kappa$ is a nilideal. This is a contradiction.

Next we consider the case (b), $e_\lambda r(N)e_\kappa=e_\lambda r(N)$ and there exist two nonzero elements c_1 and c_2 of $e_\lambda r(N)$ such that $c_1\notin c_2R$ or $c_2\notin c_1R$. Since $r(N)e_\kappa$ is completely reducible, the isomorphism θ: $Re_\lambda c_1\rightarrow Re_\lambda c_2$ can be extended to an automorphism Θ of the socle $r(N)e_\kappa$ of Re_κ. Let $\overline{\Theta}$ be an extension of Θ and a subideal Se_κ of Re_κ the domain of $\overline{\Theta}$. Further, assume that $\overline{\Theta}$ is maximal. Then $Se_\kappa\neq Re_\kappa$, for otherwise $\overline{\Theta}$ should be obtained by the multiplication of a regular element r of $e_\kappa Re_\kappa$ on the right hand and $c_1r=c_2$ and $c_2r^{-1}=c_1$, but this contradicts $c_1\notin c_2R$ or $c_2\notin c_1R$,

Let us denote by M the left R-module $\sum_{\rho\neq\kappa}\oplus Re_\rho\oplus\mathrm{Int}_{\overline{\Theta}}(Re_\kappa, Re_\kappa)$. Then, since M is finitely generated, faithful, it follows that M is a generator and the trace ideal of M is R. There exist similarly as in case (a) a family $\{x_i\}$ of elements of M and a family $\{\phi_i\}$ of left R-homomorphisms ϕ_i: $M\rightarrow R$ such that $\sum_{i=1}^n\phi_i(x_i)=1$ and ϕ_i can be expressed as follows: $\phi_i=\sum_{\rho\neq\kappa}\phi_{\rho,i}+\phi_{(\kappa,\kappa),i}$, where $\phi_{\rho,i}$, $\phi_{(\kappa,\kappa),i}\in\mathrm{Hom}_R(M, R)$, $\phi_{\rho,i}(Re_{\rho'})=0$ for $\rho'\neq\rho$, κ, $\phi_{\rho,i}(\mathrm{Int}_{\overline{\Theta}}(Re_\kappa, Re_\kappa))=0$ and $\phi_{(\kappa,\kappa),i}(Re_\rho)=0$. To $\phi_{\rho,i}$ and $\phi_{(\kappa,\kappa),i}$ there exist an element $e_\rho a_{\rho,i}$ of R and a pair of elements $e_\kappa a_{\kappa,i}$ and $e_\kappa b_{\kappa,i}$ of R such that $\phi_{\rho,i}(x_\rho)=x_\rho e_\rho a_{\rho,i}$ for all $x_\rho\in Re_\rho$ and $\phi_{(\kappa,\kappa),i}(x_\kappa\alpha+y_\kappa\beta)=x_\kappa e_\kappa a_{\kappa,i}+y_\kappa e_\kappa b_{\kappa,i}$ for all $x_\kappa\in Re_\kappa$ and for all $y_\kappa\in Re_\kappa$. Here it is to be noted that $se_\kappa a_{\kappa,i}=\overline{\Theta}(se_\kappa)e_\kappa b_{\kappa,i}$ for all $se_\kappa\in Se_\kappa$. Then, putting $x=\sum_{\rho\neq\kappa}r_{\rho,i}e_\rho+u_{\kappa,i}e_\kappa\alpha+v_{\kappa,i}e_\kappa\beta$, $r_{\rho,i}$, $u_{\kappa,i}$, $v_{\kappa,i}\in R$, we have $\sum_i e_\kappa u_{\kappa,i}e_\kappa a_{\kappa,i}e_\kappa+\sum_i e_\kappa v_{\kappa,i}e_\kappa b_{\kappa,i}e_\kappa\equiv e \bmod e_\kappa N^2e_\kappa$ and at least an element of the set $\{e_\kappa a_{\kappa,i}e_\kappa$ and $e_\kappa b_{\kappa,j}e_\kappa$, $i, j=1, 2, \cdots, n\}$ is regular in $e_\kappa Re_\kappa$. First suppose that $e_\kappa b_{\kappa,j}e_\kappa$ be regular. Then $\overline{\Theta}(se_\kappa)=se_\kappa a_{\kappa,j}(e_\kappa b_{\kappa,j}e_\kappa)^{-1}$. Let us denote by Γ the endomorphism of Re_κ which is obtained by the multiplication of $e_\kappa a_{\kappa,j}e_\kappa(e_\kappa b_{\kappa,j}e_\kappa)^{-1}$ on the right hand. Then Γ is clearly an extension of $\overline{\Theta}$. This is a contradiction. Suppose, next, $e_\kappa a_{\kappa,i}e_\kappa$ be regular. Then $se_\kappa=\overline{\Theta}(se_\kappa)e_\kappa b_{\kappa,i}(e_\kappa a_{\kappa,i}e_\kappa)^{-1}$ and the endomorphism of Re_κ which is obtained by the multiplication of $e_\kappa b_{\kappa,i}(e_\kappa a_{\kappa,i}e_\kappa)^{-1}$ is an extension of $\overline{\Theta}^{-1}$ and we arrive again at a similar contradiction. Thus, from the argument for the cases (a) and (b) we know that $e_\lambda r(N)$ is either simple or zero.

Now let us denote by $\Pi(\kappa)$ the set of primitive idempotents e_ρ such that $e_\rho r(N)e_\kappa \neq 0$. Then, since $r(N)e_\kappa \neq 0$ for every κ, $\Pi(\kappa)$ is nonempty. If $\Pi(\kappa)\cap\Pi(\mu)$ is not empty for $\kappa\neq\mu$, then there exists a primitive idempotent e_ρ such that $e_\rho r(N)e_\kappa\neq 0$, $e_\rho r(N)e_\mu \neq 0$. This contradicts the conclusion for the case (a). Thus $\Pi(\kappa)$ consists of only one primitive idempotent which we shall denote by $e_{\pi(\kappa)}$. The set $\{e_{\pi(\kappa)}\}$, for all κ, forms the set of all primitive idempotent e_λ's. Hence for every primitive idempotent e_λ of R there exists a primitive idempotent e_κ such that $e_\lambda r(N)\cong \bar{e}_\kappa \bar{R}$, where $\bar{R}=R/N$. Therefore $e_\lambda r(N)$ is a nonzero ideal and (1) of this proposition holds good.

To prove (2), suppose $r(N)e_\kappa$ is not simple. Then, since $r(N)e_\kappa$ is completely reducible, there exists a nonzero endomorphism γ such that $\operatorname{Ker}\gamma\neq 0$. Let us denote by M the left R-module $Re_\kappa/\operatorname{Ker}\gamma$ and by $\bar{\gamma}$ the isomorphism of $r(N)e_\kappa/\operatorname{Ker}\gamma$ onto $\operatorname{Im}\gamma$ which is naturally induced by γ. Clearly, $\sum_{\rho\neq\kappa}\oplus Re_\rho \oplus \operatorname{Int}_{\bar{\gamma}}(M, Re_\kappa)$ is finitely generated, faithful and hence a generator. Thus, similarly as in the preceding proof, there exist elements $e_\rho a_{\rho,i}$, $e_\kappa a_{\kappa,i}$, $e_\kappa b_{\kappa,i}$ such that (i) $v_{\kappa,i}e_\kappa a_{\kappa,i}=0$ for all elements $v_{\kappa,i}e_\kappa$ of $\operatorname{Ker}\gamma$, (ii) $z_{\kappa,i}e_\kappa a_{\kappa,i}=\gamma(z_{\kappa,i}e_\kappa)e_\kappa b_{\kappa,i}$ for all elements $z_{\kappa,i}e_\kappa$ of $r(N)e_\kappa$, and it holds $\sum_{\rho\neq\kappa}\sum_i r_{\rho,i}e_\rho a_{\rho,i} + \sum_i x_{\kappa,i}e_\kappa a_{\kappa,i} + \sum_i y_{\kappa,i}e_\kappa b_{\kappa,i}=1$, where $r_{\rho,i}$, $x_{\kappa,i}$, $y_{\kappa,i}\in R$. Hence $\sum_i e_\kappa x_{\kappa,i}e_\kappa a_{\kappa,i}e_\kappa + \sum_i e_\kappa y_{\kappa,i}e_\kappa b_{\kappa,i}e_\kappa \equiv e_\kappa \mod N^2$. However, by (i) it is known that $e_\kappa a_{\kappa,i}e_\kappa$ is a nonregular element of $e_\kappa Re_\kappa$. Hence, for some i, $e_\kappa b_{\kappa,i}e_\kappa$ is a regular element of $e_\kappa Re_\kappa$, and by (ii) $\gamma(z_{\kappa,i}e_\kappa) = z_{\kappa,i}e_\kappa a_{\kappa,i}e_\kappa(e_\kappa b_{\kappa,i}e_\kappa)^{-1}$. On the other hand, by (1) of this proposition $r(N)\subseteq l(N)$ and $e_\kappa a_{\kappa,i}e_\kappa\in N$ and hence $\gamma(z_{\kappa,i}e_\kappa)=0$ for all $z_{\kappa,i}e_\kappa\in r(N)e_\kappa$. This contradicts that γ is a nonzero endomorphism of $r(N)e_\kappa$. (2) of this proposition follows. This completes the proof.

Following F. Kasch we shall say that R is a left S-ring if $l(J)\neq 0$ for any right ideal J with $J\neq R$.

COROLLARY 2.2. *Let R be a right perfect ring. If every finitely generated, faithful left R-module is a generator, then R is a left S-ring.*

PROOF. Let J be a right ideal of R such that $J\neq R$. Denote by M the maximal right ideal containing J. Then for some κ, $R/M\cong\bar{e}_\kappa\bar{R}$, where $\bar{R}=R/N$. In the proof of (1) of Proposition 2.1, we have shown that there is an idempotent e_λ such that $e_\lambda r(N)\cong\bar{e}_\kappa\bar{R}$, and hence there is a monomorphism of R/M into R. This implies that there is a nonzero element r of R which annihilates M. Thus $0\neq r\in l(M)\subseteq l(J)$.

Now we shall introduce a version of Azumaya's lemma [1, Lemma 1].

LEMMA 2.3. *Let R be a ring (not necessarily right perfect ring) and X a left R-module. If $X=M_1\oplus M_2$ and $X=N_1\oplus N_2$ are two direct sum decompositions of X, and the left R-endomorphism ring of N_1 is completely primary, then either M_1 or M_2 has a direct summand which is isomorphic to N_1.*

PROOF. For suitable idempotents f_1, f_2, e_1 and e_2 in the R-endomorphism ring E of X (considered as the right operator domain of X), we can assume that $M_1=Xf_1$, $M_2=Xf_2$, $N_1=Xe_1$, $N_2=Xe_2$, $f_1+f_2=1$ and e_1 is a primitive idempotent of E. Since the R-endomorphism ring of N_1 is completely primary and $e_1=e_1f_1e_1+e_1f_2e_1$, either $e_1f_1e_1$ or $e_1f_2e_1$ must induce an automorphism of N_1, or what is the same, N_1 is mapped by f_1 or f_2 isomorphically upon $\tilde{N}_1$, and by e_1, $\tilde{N}_1$ is carried isomorphically onto N_1. Since N_2 is the kernel of e_1, the latter fact implies the following direct sum decomposition of X: $X=\tilde{N}_1\oplus N_2$. However, according as $e_1f_1e_1$ is regular or not, we can assume $\tilde{N}_1=Xe_1f_1\subset Xf_1$ or $\tilde{N}_1=Xe_1f_2\subset Xf_2$. Hence the conclusion follows from the above decomposition of X.

PROPOSITION 2.4. *Let R be a right perfect ring. If every finitely generated faithful left R-module is a generator, then R is an injective left R-module.*

PROOF. Let Q be the injective hull of any primitive ideal Re_κ. Assume that Re_κ is contained properly in Q. Then there exists an element u of Q such that $Ru \not\subset Re_\kappa$. Consider the left R-module $\sum_{\rho\neq\kappa}\oplus Re_\rho\oplus Ru+Re_\kappa$. It is obvious that this module is finitely generated, faithful, and hence a generator. It follows that $[\sum_{\rho\neq\kappa}\oplus Re_\rho\oplus Ru+Re_\kappa]^n=\sum_{\rho\neq\kappa}\oplus Re_\rho\oplus Re_\kappa\oplus U$, where U is a left R-module and $[\sum_{\rho\neq\kappa}\oplus Re_\rho\oplus Ru+Re_\kappa]^n$ is a direct sum of n-copies of $\sum_{\rho\neq\kappa}\oplus Re_\rho\oplus Ru+Re_\kappa$. Then, since $Re_\rho\not\cong Re_\kappa$ and endomorphism ring of Re_κ is completely primary, it follows by Lemma 2.3 that $Ru+Re_\kappa\cong Re_\kappa\oplus U'$ for a left R-module U'. However, by (2) of Proposition 2.1 the socles of Re_κ and $Ru+Re_\kappa$ respectively are simple left R-modules. Therefore $U'=0$ and $Ru+Re_\kappa\cong Re_\kappa$. Hence Re_κ is considered as a finitely generated, projective submodule of a projective left R-module $Ru+Re_\kappa$. On the other hand, by Corollary 2.2 R is a left S-ring and by Bass's theorem ([**3**, Theorem 5.4], cf. also [**3**, Theorem 1]), $Ru+Re_\kappa$ is a direct sum of Re_κ and nonzero left R-module V. But this contradicts that the socles of $Ru+Re_\kappa$ and Re_κ are the same. Hence for every primitive idempotent e_κ, Re_κ is injective.

Now we have the following main result.

THEOREM 2.5. *Let R be a right perfect ring. In order that every* *finitely generated, faithful left R-module be a generator it is necessary and sufficient that R is left self-injective and a direct sum of primitive left ideals each of which contains a minimal left ideal.*

PROOF. The necessity follows from Proposition 2.4 and (2) of Proposition 2.1. On the other hand, it will be seen that the sufficiency is obtained by Azumaya's argument of [2, Theorem 6].

It is well known that a left Artinian ring is right perfect and a left Artinian ring is quasi-Frobenius if and only if it is injective as a left R-module. Thus we obtain

THEOREM 2.6. *Let R be a left Artinian ring. In order that every finitely generated, faithful left R-module be a generator it is necessary and sufficient that R is a quasi-Frobenius ring.*

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