EQUICONTINUITY AND n-LENGTH

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Let (M, ρ) be a pseudo-metric space. We shall obtain a necessary and sufficient condition that a collection of curves can be parametrized in such a manner that the collection of parametrizations be equicontinuous. This result can be extended to the case where ρ is a quasi-pseudo-metric. The μ-length defined here differs inessentially from that originally defined by M. Morse in [**M**]. We also use ideas of [**F**] and [**S**].

Let $I=[a, b]$. If $u \in I$, then let $I_u=[a, u]$. If (N, σ) is a pseudo-metric space and if $f\colon I \to N$, then let $\omega(f; J)=\sup\{\sigma(f(u), f(v)) \mid u, v \in J\}$ whenever J is an interval contained in I. Let $C(I)$ be the space of continuous functions on I into (M, ρ) metrized by σ where $\sigma(x, y) =\sup\{\rho(x(u), y(u)) \mid u \in I\}$.

For each positive integer n and $x \in C(I)$, let

$$\mu_n x = \sup \left\{ \sum_{i=1}^{n} \rho(x(t_{i-1}), x(t_i)) \;\middle|\; a \leqq t_0 \leqq t_1 \leqq \cdots \leqq t_n \leqq b \right\}.$$

Since x is continuous and I is compact, the supremum is a maximum. Evidently $\mu_n \leqq \mu_{n+1}$, $\mu_n \leqq n\mu_1$ and $|\mu_n x - \mu_n y| \leqq 2n\sigma(x, y)$.

LEMMA 1. *Let $A \subset C(I)$ be equicontinuous and $M_n = \sup\{\mu_n x \mid x \in A\}$. The* $\lim_{n\to\infty} n^{-1}M_n = 0$.

PROOF. Let $\epsilon > 0$. There exists $\delta > 0$ such that $\omega(x; J) < \epsilon$ whenever length $J = |J| < \delta$. Let $K > \delta^{-1}|I|$ and let T_j, $j=1, 2, \cdots, K-1$, be points in I which divide I into K equal subintervals. Let $t_i \in I$, i$=0, 1, \cdots, n$, and let $\{\sigma_m\}$, $m=0, 1, \cdots, R=n+K-1$, be a nondecreasing arrangement of $\{t_i\} \cup \{T_j\}$. If $x \in A$ then

$$\sum_{i=1}^{n} \rho(x(t_{i-1}), x(t_i)) \leqq \sum_{m=1}^{R} \rho(x(\sigma_{m-1}), x(\sigma_m)) < (n + K - 1)\epsilon.$$

If $x \in C(I)$ and $u \in I$ let $\phi_{n,x}(u) = \mu_n x \mid I_u$.

LEMMA 2. *If $n > 2$, then* $\phi_{n-2,x} \geqq (1-3/n)\phi_{n,x}$.

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PROOF. It is sufficient to show that $\mu_{n-2}x \geqq (1-3/n)\mu_n x$. There exists a nondecreasing sequence $\{t_i\}$ in I such that $\sum_{i=1}^n a_i = \mu_n x$, where $a_i = \rho(x(t_{i-1}), x(t_i))$. Let

$$\begin{aligned} b_i^j &= a_i \qquad \text{if } i > j+1 \quad \text{or} \quad i < j-1, \\ &= 0 \qquad \text{if } j-1 \leqq i \leqq j+1 \end{aligned}$$

for $i = 0, 1, \cdots, n$ and $j = 1, 2, \cdots, n$.

If $A = \sum_{j=1}^n \sum_{i=1}^n b_i^j$, then

$$\begin{aligned} A &= \sum_{j=1}^{n-2} \sum_{i=j+2}^{n} a_i + \sum_{j=3}^{n} \sum_{i=1}^{j-2} a_i \\ &= \sum_{i=3}^{n} \sum_{j=1}^{i-2} a_i + \sum_{i=1}^{n-2} \sum_{j=i+2}^{n} a_i \\ &= \sum_{i=3}^{n} (i-2)a_i + \sum_{i=1}^{n-2} (n-i-1)a_i \\ &= (n-3) \sum_{i=1}^{n} a_i + a_1 + a_n. \end{aligned}$$

Hence there exists k such that $\sum_{i=1}^n b_i^k \geqq (1-3/n)\mu_n x$. If t_{k-1} and t_k are deleted from $\{t_i\}$ and if the resulting nondecreasing sequence is labelled $\{\sigma_j\}$, $j = 0, 1, \cdots, n-2$, then

$$\mu_{n-2}x \geqq \sum_{j=1}^{n-2} \rho(x(\sigma_{j-1}), x(\sigma_j)) \geqq \sum_{i=1}^{n} b_i^k \geqq (1 - 3/n)\mu_n x.$$

We now define a μ-length on $C(I)$ by $\mu = \sum_{n=1}^\infty 2^{-n}\mu_n$. Thus $\mu_1 \leqq \mu \leqq \mu_1 \sum_{n=1}^\infty 2^{-n} n = 2\mu_1$. If $x \in C(I)$, let $\phi_x(u) = \mu x | I_u$. Evidently $2^{-k}\omega(\phi_{k,x}; J) \leqq \omega(\phi_x; J) \leqq 2\omega(x; J)$.

LEMMA 3. $\omega(x, J) \leqq \max\{6n^{-1}\mu_n x, 2^{n+1}\omega(\phi_x; J)\}$ *for all* n.

PROOF. Let $u, v \in J$ with $u < v$ and let $2\eta = \rho(x(u), x(v))$. If $\eta > 3n^{-1}\mu_n x$, then

$$\phi_{n,x}(v) - \phi_{n,x}(u) \geqq \phi_{n-2,x}(u) + 2\eta - \phi_{n,x}(u) \geqq 2\eta - 3n^{-1}\phi_{n,x}(u) > \eta.$$

Hence $\eta < \omega(\phi_{n,x}; J) \leqq 2^n \omega(\phi_x; J)$.

COROLLARY. *A necessary and sufficient condition that* ϕ_x *be constant on* J *is that* x *be constant on* J.

Let $C_\mu = \{X \in C([0, 1]) | \phi_X(u) = (\mu X)u \text{ for all } u \in [0, 1]\}$. Evidently $\omega(\phi_X; J) = (\mu X)|J|$ and

$$2^{-1}(\mu X)|J| \leqq \omega(X;J) \leqq \max\{6n^{-1}(\mu_n X),\ 2^{n+2}(\mu_1 X)|J|\}$$

for all n whenever $X \in C_\mu$.

LEMMA 4. *Let $A \subset C_\mu$ and $M_n = \sup\{\mu_n X | X \in A\}$. Then A is equicontinuous if and only if $\lim_{n\to\infty} n^{-1}M_n = 0$.*

PROOF. Choose $\epsilon > 0$ and take j so that $6M_j < j\epsilon$. Let $\delta = [2^{j+2}M_1]^{-1}\epsilon$. If $u \leqq v < u + \delta$, then

$$\rho(X(u), X(v)) \leqq \omega(X; [u, v]) \leqq \max\{6j^{-1}M_j,\ 2^{j+2}M_1\delta\} = \epsilon$$

and the proof is complete because of Lemma 1.

LEMMA 5. *Let $X_n \in C_\mu$. Then $\lim_{n\to\infty} n^{-1}(\mu_n X_n) = 0$ if and only if $\{X_n\}$ is equicontinuous.*

PROOF. If $\{X_n\}$ is equicontinuous, then

$$0 = \lim_{k\to\infty} k^{-1} \sup_n \mu_k X_n \geqq \lim_{k\to\infty} k^{-1}(\mu_k X_k) \geqq 0.$$

The proof in the other direction is like that of the preceding lemma.

If $x \in C(I)$ and $y \in C(J)$, let $D_F(x, y) = \inf\ \{\sigma(x, y \circ h) |\ h$ is a sense-preserving homeomorphism of I onto $J\}$. Then D_F is a pseudo-metric and is a metric on the space of Fréchet equivalence classes: $x\ F\ y$ if $D_F(x, y) = 0$. Such an equivalence class is a Fréchet curve. If m is continuous and monotonically nondecreasing from I onto J then $z\ F\ x$ where z is defined by $x = z \circ m$. It is easy to see that $|\mu_n x - \mu_n y| \leqq 2n\ D_F(x, y)$ and $|\mu_x - \mu_y| \leqq 4D_F(x, y)$. If ξ and η are Fréchet curves with $x \in \xi$ and $y \in \eta$, then let $\mu_n\xi = \mu_n x$, $\mu\xi = \mu x$ and $D_F(\xi, \eta) = D_F(x, y)$.

LEMMA 6. *If X, $Y \in C_\mu$ and $X\ F\ Y$, then $X = Y$.*

PROOF. There exist homeomorphisms $\{h_k\}$ such that $\sigma(X, Y \circ h_k) < k^{-1}$. If $u \in [0, 1]$ then $(\mu X)|u - h_k(u)| = |\phi_X(u) - \phi_Y(h_k(u))| < 4k^{-1}$ so that h_k converges uniformly to the identity on $[0, 1]$.

If $x \in C(I)$, let $X = x \circ \phi x/\mu x$ if x is not constant, and let $X = x(a)$ otherwise. Thus, by some earlier remarks, $X\ F\ x$.

LEMMA 7. *If $x \in C(I)$, then $X \in C_\mu$.*

PROOF. If $u \in [0, 1]$, there exists $c \in I$ such that $(\mu x)u = \phi_x(c)$. Thus $(X|\ [0, u])\ F\ (x|\ I_c)$ so that $\phi_x(u) = \phi_x(c) = (\mu x)u = (\mu X)u$.

It follows that each Fréchet curve has a unique representation in C_μ.

Let F be the space of Fréchet curves metrized by D_F. If x, $y \in \xi \in F$ then range $x =$ range y. Let $[\xi]$ denote range x.

THEOREM. *Let K be a compact subset of M and $B\subset F$. If $[\xi]\subset K$ for all $\xi\in B$ then B is sequentially compact if and only if* $\lim_{n\to\infty} n^{-1}M_n=0$ *where* $M_n=\sup\{\mu_n\xi \mid \xi\in B\}$. *If $[\xi_n]\subset K$, then $\{\xi_n\}$ contains a convergent subsequence if and only if* $\lim_{n\to\infty} n^{-1}(\mu_n\xi_n)=0$.

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