EQUICONTINUITY AND \( n \)-LENGTH

EDWARD SILVERMAN

Let \((M, \rho)\) be a pseudo-metric space. We shall obtain a necessary and sufficient condition that a collection of curves can be parametrized in such a manner that the collection of parametrizations be equicontinuous. This result can be extended to the case where \(\rho\) is a quasi-pseudo-metric. The \(\mu\)-length defined here differs inessentially from that originally defined by M. Morse in [M]. We also use ideas of [F] and [S].

Let \(I = [a, b]\). If \(u \in I\), then let \(I_u = [a, u]\). If \((N, \sigma)\) is a pseudo-metric space and if \(f: I \to N\), then let \(\omega(f; J) = \sup \{\sigma(f(u), f(v)) | u, v \in J\}\) whenever \(J\) is an interval contained in \(I\). Let \(C(I)\) be the space of continuous functions on \(I\) into \((M, \rho)\) metrized by \(\sigma\) where \(\sigma(x, y) = \sup \{\rho(x(u), y(u)) | u \in I\}\).

For each positive integer \(n\) and \(x \in C(I)\), let
\[
\mu_n x = \sup \left\{ \frac{1}{n} \sum_{i=1}^{n} \rho(x(t_{i-1}), x(t_i)) \mid a \leq t_0 \leq t_1 \leq \cdots \leq t_n \leq b \right\}.
\]

Since \(x\) is continuous and \(I\) is compact, the supremum is a maximum. Evidently \(\mu_n x \leq \mu_{n+1} x\), \(\mu_n x \leq n \mu_1\) and \(|\mu_n x - \mu_n y| \leq 2 n \sigma(x, y)\).

**Lemma 1.** Let \(A \subset C(I)\) be equicontinuous and \(M_n = \sup \{\mu_n x \mid x \in A\}\). The \(\lim_{n \to \infty} n^{-1} M_n = 0\).

**Proof.** Let \(\varepsilon > 0\). There exists \(\delta > 0\) such that \(\omega(x; J) < \varepsilon\) whenever length \(J = |J| < \delta\). Let \(K > \delta^{-1} |I|\) and let \(T_j, j = 1, 2, \ldots, K - 1\), be points in \(I\) which divide \(I\) into \(K\) equal subintervals. Let \(t_i \in I, i = 0, 1, \ldots, n\), and let \(\{\sigma_m\}, m = 0, 1, \ldots, R = n + K - 1\), be a non-decreasing arrangement of \(\{t_i\} \cup \{T_j\}\). If \(x \in A\) then
\[
\sum_{i=1}^{n} \rho(x(t_{i-1}), x(t_i)) \leq \sum_{m=1}^{R} \rho(x(\sigma_{m-1}), x(\sigma_m)) < (n + K - 1) \varepsilon.
\]

If \(x \in C(I)\) and \(u \in I\) let \(\phi_{n,x}(u) = \mu_n x \mid I_u\).

**Lemma 2.** If \(n > 2\), then \(\phi_{n-2,x} \geq (1 - 3/n) \phi_{n,x}\).

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PROOF. It is sufficient to show that $\mu_{n-2}x \geq (1 - 3/n)\mu_n x$. There exists a nondecreasing sequence $\{t_i\}$ in $I$ such that $\sum_{i=1}^{n} a_i = \mu_n x$, where $a_i = \rho(x(t_{i-1}), x(t_i))$. Let
\[
{b^j_i} = \begin{cases} 
  a_i & \text{if } i > j + 1 \quad \text{or} \quad i < j - 1, \\
  0 & \text{if } j - 1 \leq i \leq j + 1 
\end{cases}
\]
for $i = 0, 1, \ldots, n$ and $j = 1, 2, \ldots, n$.

If $A = \sum_{i=1}^{n} \sum_{j=1}^{n} b^j_i$, then
\[
A = \sum_{i=3}^{n} \sum_{j=i+2}^{n} a_i + \sum_{j=3}^{n} \sum_{i=1}^{j-2} a_i
= \sum_{i=3}^{n} \sum_{j=1}^{i} a_i + \sum_{j=1}^{n} \sum_{i=1}^{j} a_i
= \sum_{i=3}^{n} (i - 2)a_i + \sum_{i=1}^{n} (n - i - 1)a_i
= (n - 3) \sum_{i=1}^{n} a_i + a_1 + a_n.
\]
Hence there exists $k$ such that $\sum_{i=1}^{n} b^k_i \geq (1 - 3/n)\mu_n x$. If $t_{k-1}$ and $t_k$ are deleted from $\{t_i\}$ and if the resulting nondecreasing sequence is labelled $\{\sigma_j\}, j = 0, 1, \ldots, n - 2$, then
\[
\mu_{n-2}x \geq \sum_{j=1}^{n-2} \rho(x(\sigma_{j-1}), x(\sigma_j)) \geq \sum_{i=1}^{n} b^k_i \geq (1 - 3/n)\mu_n x.
\]

We now define a $\mu$-length on $C(I)$ by $\mu = \sum_{n=1}^{\infty} 2^{-n} \mu_n$. Thus $\mu_1 \leq \mu \leq \mu_1 \sum_{n=1}^{\infty} 2^{-n} n = 2\mu_1$. If $x \in C(I)$, let $\phi_x(u) = \mu x|I_u$. Evidently $2^{-k}\omega(\phi_{k,x}; J) \leq \omega(\phi_x; J) \leq 2\omega(x; J)$.

**Lemma 3.** $\omega(x, J) \leq \max \{6n^{-1}\mu_n x, 2n^{+1}\omega(\phi_x; J)\}$ for all $n$.

**Proof.** Let $u, v \in J$ with $u < v$ and let $2\eta = \rho(x(u), x(v))$. If $\eta > 3n^{-1}\mu_n x$, then
\[
\phi_{n,x}(v) - \phi_{n,x}(u) \geq \phi_{n-2,x}(u) + 2\eta - \phi_{n,x}(u) \geq 2\eta - 3n^{-1}\phi_{n,x}(u) > \eta.
\]
Hence $\eta < \omega(\phi_{n,x}; J) \leq 2n\omega(\phi_x; J)$.

**Corollary.** A necessary and sufficient condition that $\phi_x$ be constant on $J$ is that $x$ be constant on $J$.

Let $C_\mu = \{X \in C([0, 1]) | \phi_X(u) = (\mu X)u \text{ for all } u \in [0, 1] \}$. Evidently $\omega(\phi_x; J) = (\mu X)|J|$ and
2^{-1}(\mu X)\lfloor J \rceil \leq \omega (X; J) \leq \max \{ 6n^{-1}(\mu_n X), 2^{n+2}(\mu_1 X)\lfloor J \rceil \}

for all \( n \) whenever \( X \in C_\mu \).

**Lemma 4.** Let \( A \subset C_\mu \) and \( M_n = \sup \{ \mu_n X \mid X \in A \} \). Then \( A \) is equi-
continuous if and only if \( \lim_{n \to \infty} n^{-1}M_n = 0 \).

**Proof.** Choose \( \epsilon > 0 \) and take \( j \) so that \( 6M_j < j\epsilon \). Let \( \delta = [2^{j+2}M_1]^{-1}\epsilon \).
If \( u \leq v < u + \delta \), then

\[
\rho(X(u), X(v)) \leq \omega(X; [u, v]) \leq \max \{ 6j^{-1}M_j, 2^{j+2}M_1\delta \} = \epsilon
\]
and the proof is complete because of Lemma 1.

**Lemma 5.** Let \( X_n \in C_\mu \). Then \( \lim_{n \to \infty} n^{-1}(\mu_n X_n) = 0 \) if and only if
\( \{ X_n \} \) is equicontinuous.

**Proof.** If \( \{ X_n \} \) is equicontinuous, then

\[
0 = \lim_{k \to \infty} k^{-1} \sup \mu_k X_n \geq \lim_{k \to \infty} k^{-1}(\mu_k X_k) \geq 0.
\]

The proof in the other direction is like that of the preceding lemma.

If \( x \in C(I) \) and \( y \in C(J) \), let \( D_F(x, y) = \inf \{ \sigma(x, y \circ h) \mid h \text{ is a sense-preserving homeomorphism of } I \text{ onto } J \} \). Then \( D_F \) is a pseudo-metric and is a metric on the space of Fréchet equivalence classes: \( x \sim y \) if \( D_F(x, y) = 0 \). Such an equivalence class is a Fréchet curve. If \( m \) is continuous and monotonically nondecreasing from \( I \) onto \( J \) then \( z \sim x \) where \( z \) is defined by \( x = z \circ m \). It is easy to see that \( \| \mu_n x - \mu_n y \| \leq 2n \| D_F(x, y) \| \text{ and } \| \mu_x - \mu_y \| \leq 4D_F(x, y) \).

**Lemma 6.** If \( X, Y \in C_\mu \) and \( X \sim Y \), then \( X = Y \).

**Proof.** There exist homeomorphisms \( \{ h_k \} \) such that \( \sigma(X, Y \circ h_k) < k^{-1} \). If \( u \in [0, 1] \) then \( (\mu X)\lfloor u - h_k(u) \rceil = | \phi_X(u) - \phi_Y(h_k(u)) | < k^{-1} \) so that \( h_k \) converges uniformly to the identity on \( [0, 1] \).

If \( x \in C(I) \), let \( X = x \circ \phi_x / \mu x \) if \( x \) is not constant, and let \( X = x(a) \) otherwise. Thus, by some earlier remarks, \( X \sim x \).

**Lemma 7.** If \( x \in C(I) \), then \( X \in C_\mu \).

**Proof.** If \( u \in [0, 1] \), there exists \( c \in I \) such that \( (\mu x)u = \phi_x(c) \). Thus \( (X \lfloor [0, u]) \sim F(x \lfloor I_c) \) so that \( \phi_x(u) = \phi_x(c) = (\mu x)u = (\mu X)u \).

It follows that each Fréchet curve has a unique representation in \( C_\mu \).

Let \( F \) be the space of Fréchet curves metrized by \( D_F \). If \( x, y \in \xi \in F \) then range \( x = \text{range } y \). Let \( [\xi] \) denote range \( x \).
**Theorem.** Let $K$ be a compact subset of $M$ and $B \subseteq F$. If $\{\xi\} \subseteq K$ for all $\xi \in B$ then $B$ is sequentially compact if and only if $\lim_{n \to \infty} n^{-1}M_n = 0$ where $M_n = \sup \{ \mu_n \xi | \xi \in B \}$. If $\{\xi_n\} \subseteq K$, then $\{\xi_n\}$ contains a convergent subsequence if and only if $\lim_{n \to \infty} n^{-1}(\mu_n \xi_n) = 0$.

**References**


Purdue University