PROPERTIES OF LUSIN SETS WITH APPLICATIONS TO PROBABILITY

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In this note we shall show that a Lusin set is so thin that it can be used to construct several illuminating examples for probability.

I. **Notation.** In this paper a Lusin set will be understood to be an uncountable subset Ω of the interval $I=[0, 1]$ such that if F is a first category subset of I, then $\Omega\cap F$ is, at most, a countable set or, equivalently, if $\{y_i\}$ is a countable dense subset of I and $\{\epsilon_i\}$ is a sequence of positive numbers, then $\Omega-\bigcup N(y_i, \epsilon_i)$ is countable, where $N(y, \epsilon) =I\cap(y-\epsilon/2, y+\epsilon/2)$. By a universally measurable set we shall mean a subset E of R^1 such that if μ is a nonatomic probability measure on the Borel subsets $\mathcal{B}_0$ of R^1, then $\mu_*(E)=\mu^*(E)$. A set E is universally measurable if, and only if, every homeomorphism ϕ of R^1 carries E onto a set $\phi(E)$ which is Lebesgue measurable. A universal null set is a universally measurable set which is mapped into a Lebesgue null set by all homeomorphisms. Since it imposes no real restriction to suppose that a probability measure on $\mathcal{B}_0$ is nonatomic, henceforth we do so suppose. Finally, denote by $\mathcal{B}$ the Borel subsets $\Omega\cap B$, $B\in\mathcal{B}_0$, of Ω, i.e., $\mathcal{B}=\mathcal{B}_0|\Omega$.

II. The Borel measurable images of a Lusin set Ω are universal null sets. Notice that Borel measurability has two possible interpretations in the preceding assertion.

(1) If f is a $\mathcal{B}_1$-measurable function on I, then $f(\Omega)$ is a universal null set in R^1, where $\mathcal{B}_1=\mathcal{B}_0|I$.

(2) If f is a $\mathcal{B}$-measurable function on Ω, then $f(\Omega)$ is a universal null set.

For continuous f these statements have long been verified (e.g. [**3**, p. 528]); but, for Borel measurable f they seem to have gone unnoticed. Toward their verification let us first recall [**3**, pp. 399–400] that there is a first category subset P of I such that $f_1=f|I_1$ is continuous where $I_1=I-P$. Let $\{y_i\}$ be a countable dense subset of I_1 and $\epsilon>0$. Then $\{y_i\}$ is dense in Ω and, recalling that μ is nonatomic, there is a sequence ϵ_i of positive numbers such that $\mu(f_1(Q_i))<\epsilon\cdot 2^{-i}$ where $Q_i=I_1\cap N(y_i, \epsilon_i)$. Let $Q=\bigcup_i Q_i$ and $P_1=I_1-Q$. The set $F=P\cup P_1$ is seen to be a first category subset of I and, hence, $F\cap\Omega$

Received bv the editors June 21, 1967 and, in revised form October, 20, 1967.

[1] This author is partially supported by an NSF grant.

is countable which implies that $\mu(f(F\cap\Omega))=0$. Moreover, since $\Omega-F\subset Q$, $f_1(Q_i)=f(Q_i)$, and $\mu(\bigcup f_1(Q_i))\leqq\sum\mu(f_1(Q_i))<\epsilon$ we have $\mu(f(\Omega-F))=0$ and assertion (1) is established. Although minor changes in notation will yield (2), perhaps it is better to put them down: There is a first category subset P of Ω ([**3**, pp. 399–400]) such that $f_1=f|I_1$ is continuous where $I_1=\Omega-P$. Let $\{y_i\}$ be a countable dense subset of I_1, $\epsilon>0$, and $\{\epsilon_i\}$ satisfy $\mu(f_1(Q_i))<\epsilon\cdot 2^{-i}$, where $Q_i=I_1\cap N(y_i,\epsilon_i)$. As before the set $F=P\cup P_1$ is a first category subset of Ω where $Q=\bigcup Q_i$ and $P_1=I_1-Q$. Hence [**3**, pp. 525–526] F is a countable set and (2) obtains.

III. **Applications to probability.** Recall [**2**] that a D-space is a measurable space $(X, \mathfrak{F})$ such that (i) $\mathfrak{F}$ is a separable sigma algebra of subsets of X and (ii) if f is a $\mathfrak{F}$-measurable function then $f(X)$ is a universally measurable set. The following characterization of a D-space has been established in [**2**]: $(X, \mathfrak{F})$ is a D-space if, and only if, $(X, \mathfrak{A}, \mu)$ is perfect (i.e., if f is $\mathfrak{A}$-measurable, $A\subset R^1$, and $f^{-1}(A)\in\mathfrak{A}$, then there exists a Borel subset B of A such that $\mu(f^{-1}(B))=\mu(f^{-1}(A))$) whenever $\mathfrak{A}$ is a separable subsigma algebra of $\mathfrak{F}$ and μ is a probability measure on $\mathfrak{A}$. If $f(X)$ is an analytic set for each $\mathfrak{F}$-measurable f, then $(X, \mathfrak{F})$ is called a Lusin space [**1**].

We shall verify the following statements which settle questions raised or implied in [**1**] and [**2**].

(3) Not every D-space is a Lusin space.

(4) The space $(\Omega, \mathfrak{B})$ is a D-space but not a Lusin space.

(5) The identity function j: $j(x)=x$, $x\in\Omega$, is a $\mathfrak{B}$-measurable function whose range is not an analytic set.

(6) If ν is a probability measure on a separable subsigma algebra $\mathfrak{F}$ of $\mathfrak{B}$, then ν is the restriction to $\mathfrak{F}$ of a probability measure on $\mathfrak{B}$.

(7) All probability measures on $\mathfrak{B}$ are atomic. If f is a $\mathfrak{B}$-measurable function on Ω, then it follows from (2) that $f(\Omega)$ is a universal null set. Because an uncountable analytic set contains a perfect set and as a result, supports a non atomic probability measure, $f(\Omega)$ contains no uncountable analytic set. In particular, the range of the identity function j is not an analytic set. Hence $(\Omega, \mathfrak{B})$ is a D-space which is not a Lusin space and (3)–(5) are established. Turning now to (6) and (7), suppose that μ is a nonatomic probability measure on $\mathfrak{F}$. Let $\{E(0), E(1)\}$ be a partition of Ω into two sets of μ-measure 2^{-1}, and let $\{E(i_1,\cdots,i_k,0), E(i_1,\cdots,i_k,1)\}$ be a partition of $E(i_1,\cdots,i_k)$ into two sets of μ-measure $2^{-(k+1)}$. Let f_k be defined on Ω by $f_k(x)=\sum_{j\leqq k} i_j(3^{-j})$ if $x\in E(i_1,\cdots,i_k)$ and, finally, let $f(x)=\lim_k f_k(x)$. Then f is a $\mathfrak{F}$-measurable function on Ω and it follows

from [2, Lemma] that there is a $\mathfrak{F}$-measurable set Ω_0 such that $\mu(\Omega_0)=1$ and $f(\Omega_0)$ is a Borel set. Because f is also $\mathfrak{B}$-measurable, $f(\Omega)$ is a universal null set. Hence $f(\Omega_0)$, being a subset of a universal null set and also a Borel set, must be countable. Let $\{t_j\}$ be an enumeration of $f(\Omega_0)$. Notice that $f(E(i_1, \cdots, i_k)) \cap f(E(j_1, \cdots, j_k)) \neq \emptyset$ if, and only if, $i_p = j_p$, $p \leqq k$. Let $E_j = E(i_1^j, \cdots, i_{j+1}^j) \supset f^{-1}(t_j)$ and let $E = \bigcup_j E_j$. Then $\mu(E) \leqq 2^{-1}$ and $f(\Omega - E) \cap f(\Omega_0) = \emptyset$. This contradiction establishes that there are no nonatomic probability measures on $\mathfrak{F}$. The decomposition of a probability measure ν on $\mathfrak{F}$ into its atomic and nonatomic parts represents ν as a convex combination of an atomic and a nonatomic probability measure. Since the nonatomic component of ν is zero, ν is atomic. Moreover, elementary measure theory tells us that each atom can be associated with a point of Ω: Suppose that $\mathfrak{F}$ is generated by the sequence $\{F_j\}$ of $\mathfrak{B}$-measurable sets and that $\nu = \sum x_i \nu_i$, where $x_i > 0$, $\sum x_i = 1$, and the ν_i's are mutually singular two-valued probability measures on $\mathfrak{F}$. Let $p_i \in \bigcap_j \{F_j, \nu_i(F_j) = 1\}$ and let λ be defined on $\mathfrak{B}$ by $\lambda(E) = \sum_{\{i; p_i \in E\}} x_i$. Then λ is an extension of ν to $\mathfrak{B}$.

IV. **Borel images and measure.** Recalling a construction of a Cantor set of positive Lebesgue measure in I reminds us that any Borel probability measure on I lies on a first category subset of I. This latter fact can be interpreted to say that if ϕ is a nondecreasing function from I into I, then there is a first category subset F of I such that the Lebesgue measure $m(\phi(I-F))=0$. The technique applied to establish (1) permits us to claim results of which the following is an instance.

(8) If b is Borel measurable on a separable metric space M, then there is a first category subset F of M such that $m(b(M-F))=0$.

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