THE ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF LINEAR THIRD ORDER DIFFERENTIAL EQUATIONS

Y. P. SINGH[1]

1. This paper is a study of asymptotic properties of solutions of the differential equation

$$y''' + p(t)y' + q(t)y = 0. \tag{L}$$

Throughout, we shall assume that $p(t)$, $p'(t)$ and $q(t)$ are continuous, and $p(t)$, $q(t)$ are bounded and do not change sign on $[a, \infty)$, $a \geqq 0$. Two theorems are provided here, and the techniques used are similar to ones used by Lazer [3], Švec [4] and Zlámal [5] in previous studies of this differential equation. The proofs are based on the following three lemmas. The first lemma is the result due to E. Esclangon [2] (for another source see [1]) and the other two lemmas are elementary and will not be proved here.

LEMMA 1.1. *Let the function $p_i(t)$, $i=0, 1, \cdots, n$ be continuous and bounded for $t \geqq t_0$. If*

$$y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y = p_0(t)$$

and $y(t)$ is bounded for $t \geqq t_0$, then its derivatives $y^{(k)}(t)$ $(1 \leqq k \leqq n)$ are also bounded for $t \geqq t_0$.

LEMMA 1.2. *Let $f(t) \in C^1[a, \infty)$. If $\int_a^\infty f^2(t)dt < \infty$ and $f'(t)$ is bounded, then $f(t) \to 0$ as $t \to \infty$.*

LEMMA 1.3. *Let $f(t) \in C^2[a, \infty)$. If $f(t) \to 0$ as $t \to \infty$ and $f''(t)$ is bounded, then $f'(t) \to 0$ as $t \to \infty$.*

I am indebted to Professor A. C. Lazer for many fruitful conversations concerning this differential equation.

2. In this section we consider the behavior of solutions of (L) subject to the conditions $p(t) \leqq 0$ and $p'(t) - 2q(t) \geqq A > 0$. We will use several times the following identity, which has played an important role in most of the previous investigations of (L). If $y(t)$ is a solution of (L) and

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$$F(y(t)) \equiv y'^2(t) - 2y(t)y''(t) - p(t)y^2(t),$$

then

$$(1) \qquad F(y(t)) = F(y(a)) - \int_a^t (p'(t) - 2q(t))y^2(t)dt.$$

From (1) it follows that if $y(t)$ is a nontrivial solution of (L) and $p'(t) - 2q(t) \geqq A > 0$, then $F(y(t))$ is strictly decreasing.

LEMMA 2.1. *Let* $p(t) \leqq 0$, $p'(t) - 2q(t) \geqq A > 0$. *If* $y(t)$ *is a solution of* (L) *for which* $F(y(t)) > 0$ *for all* $t \in [a, \infty)$, *then* $y^{(k)}(t) \to 0$ *as* $t \to \infty$, $k = 0, 1, 2, 3$.

PROOF. Since $F(y(t)) > 0$, $p'(t) - 2q(t) \geqq A > 0$, it follows from (1) that for all $t \geqq a$,

$$\int_a^t y^2(s)ds \leqq \frac{F(y(a))}{A},$$

and hence

$$\int_a^\infty y^2(s)ds < \infty.$$

We assert that $y'(t)$ is bounded. There are two possibilities.

(a) $y''(t)$ has arbitrarily large zeros.

(b) There exists a number c such that $y''(t) \neq 0$ for $t \geqq c$.

In case of possibility (a), $y'(t)$ has arbitrarily large maxima, minima. At every maxima and minima of $y'(t)$, we have

$$y'^2(t) - p(t)y^2(t) = F(y(a)) - \int_a^t (p'(t) - 2q(t))y^2(t)dt$$

or

$$y'^2(t) \leqq F(y(a)) + p(t)y^2(t) - A\int_a^t y^2(t)dt \leqq F(y(a)).$$

Thus $y'(t)$ is bounded at its maxima and minima and hence bounded on $[a, \infty)$.

In case of possibility (b), since $y''(t) \neq 0$ for $t \geqq c$, $y'(t)$ has constant sign after some t, say $t = t_1 \geqq c$, and thus either $y'(t)y''(t) < 0$ or $y'(t)y''(t) > 0$ for $t \geqq t_1$. Because $\int_a^\infty y^2(s)ds$ is convergent, $y'(t)$ and $y''(t)$ cannot have the same sign, and thus we have $y'(t)y''(t) < 0$ and from this our assertion follows.

From the boundedness of $y'(t)$ and $\int_a^\infty y^2(s)ds < \infty$ and Lemma 1.2, we conclude that $\lim_{t\to\infty} y(t)=0$. Now from Lemmas 1.1 and 1.3, it follows at once that

$$\lim_{t\to\infty} y'(t) = \lim_{t\to\infty} y''(t) = \lim_{t\to\infty} y'''(t) = 0.$$

LEMMA 2.2. *Let $p(t)\leqq 0$ and $p'(t)-2q(t)\geqq A>0$. If $z_2(t)$ is the solution of* (L) *defined by the initial conditions $z_2(a)=z_2'(a)=0$, $z_2''(a)=1$, then* $\lim_{t\to\infty} z_2(t)=\infty$.

PROOF. Since $F[z_2(a)]=0$ and $F[z_2(t)]$ is strictly decreasing, $F[z_2(t)]=z_2'^2(t)-2z_2(t)z_2''(t)-p(t)z_2^2(t)<0$ for $t>a$. Since $p(t)\leqq 0$, $z_2(t)>0$, $z_2'(t)>0$ and $z_2''(t)>0$ for $t>a$ from which the assertion follows.

THEOREM 2.3. *If $p(t)\leqq 0$ and $p'(t)-2q(t)\geqq A>0$, then there exist two independent nontrivial solutions $u(t)$ and $v(t)$ of* (L) *which tend to zero with their first three derivatives. If $y(t)$ is any nontrivial solution of* (L) *which is not a linear combination of $u(t)$ and $v(t)$, then $|y(t)|$ tends to infinity as t tends to infinity.*

PROOF. Let z_0, z_1, z_2 be the solutions of (L) satisfying the initial conditions

$$z_i^{(j)} = \delta_{ij} = 0, \quad i\neq j, \qquad i,j=0,1,2.$$
$$= 1, \quad i=j,$$

For each integer $n>a$, let b_{0n}, b_{2n} and c_{1n}, c_{2n} be numbers such that

$$\begin{aligned} b_{0n}z_0(n)+b_{2n}z_2(n) &= 0, \\ c_{1n}z_1(n)+c_{2n}z_2(n) &= 0 \end{aligned} \tag{2}$$

and

$$b_{0n}^2+b_{2n}^2 = c_{1n}^2+c_{2n}^2 = 1. \tag{3}$$

Let $u_n(t)$ and $v_n(t)$ be the nontrivial solutions of (L) defined by

$$u_n(t) = b_{0n}z_0(t)+b_{2n}z_2(t), \qquad v_n(t) = c_{1n}z_1(t)+c_{2n}z_2(t).$$

Since $u_n(n)=v_n(n)=0$, we have $F(u_n(n))\geqq 0$, $F(v_n(n))\geqq 0$. Because $F(y(t))$ is a decreasing function, it follows that

$$F(u_n(t))>0, \qquad F(v_n(t))>0 \quad \text{for } t\in[a,n). \tag{4}$$

Now by (3) there exists a sequence of integers $\{n_j\}$ such that the sequences $\{b_{0n_j}\}$, $\{b_{2n_j}\}$ and $\{c_{1n_j}\}$, $\{c_{2n_j}\}$ converge respectively to numbers b_0, b_2, c_1 and c_2 such that

(5) $$b_0^2 + b_2^2 = c_1^2 + c_2^2 = 1.$$

Let $u(t)$, $v(t)$ be the solutions of (L) defined by

(6) $$u(t) = b_0 z_0(t) + b_2 z_2(t), \qquad v(t) = c_1 z_1(t) + c_2 z_2(t).$$

By the linear independence of z_0, z_1 and z_2 and from (5), it follows that u and v are nontrivial solutions of (L). Clearly the sequences $\{u_{n_j}^{(k)}(t)\}$ and $\{v_{n_j}^{(k)}(t)\}$ converge to $u^{(k)}(t)$ and $v^{(k)}(t)$, $k=0, 1, 2, 3$, on $[a, \infty)$ respectively, and from (4) it follows that $F(u(t)) \geqq 0$ and $F(v(t)) \geqq 0$ for all $t \in [a, \infty)$. Hence, by Lemma 2.1, $u^{(k)}(t)$, $v^{(k)}(t) \to 0$ as $t \to \infty$, $k=0, 1, 2, 3$.

If $u(t)$ and $v(t)$ were dependent, then from (5) it would follow that $u(t) = K z_2(t)$, for some $K \neq 0$. Since $z_2(a) = z_2'(a) = 0$ and $z_2''(a) = 1$, we have

$$F(u(t)) = -\int_a^t (p'(t) - 2q(t))u^2(t)dt < 0 \qquad \text{for } t > a,$$

which is contradictory. Thus $u(t)$ and $v(t)$ are independent.

Consider the solution $u(t)$, $v(t)$ and $z_2(t)$. Now there are two possibilities—that either $u(t)$, $v(t)$ and $z_2(t)$ are dependent or that they are independent. Suppose these are dependent. We can find numbers B, C and D, not all zero such that

$$w(t) \equiv Bu(t) + Cv(t) + Dz_2(t) \equiv 0.$$

By the independence of $u(t)$ and $v(t)$, $D \neq 0$. By Lemma 2.2, we have $\lim_{t\to\infty} |z_2(t)| = \infty$, and thus when $t \to \infty$, $w(t) \to \infty$, which is not possible. Hence $u(t)$, $v(t)$ and $z_2(t)$ are independent, and since the order of (L) is three, it follows from the theory of linear differential equations that every solution $y(t)$ of (L) is of the form $y(t) = a\,u(t) + b\,v(t) + c\,z_2(t)$, for some constants a, b and c. Since $\lim_{t\to\infty} |z_2(t)| = \infty$, $y(t) \to 0$ as $t \to \infty$ if and only if $c=0$. From this, assertions of the theorem follow immediately.

3. We will now consider the behavior of the solutions of (L) subject to the conditions $p(t) \leqq 0$, and $2q(t) - p'(t) \geqq d > 0$. Under these conditions, if $y(t)$ is any nontrivial solution of (L), then the function $F(y(t))$ is strictly increasing.

LEMMA 3.1. *If $p(t) \leqq 0$, $2q(t) - p'(t) \geqq d > 0$ and $y(t)$ is any solution of* (L) *satisfying the initial conditions*

$$y(c) = 0, \qquad y'(c) = 0 \quad \textit{and} \quad y''(c) > 0,$$

(*where c is an arbitrary number greater than a*), *then*

$$y(t) > 0, \qquad y'(t) < 0, \qquad y''(t) > 0 \quad \textit{for all } t \in [a, c).$$

PROOF. Since $y(c)=0$, $y'(c)=0$ and $F(y(t))$ is an increasing function, we have $F(y(t))<0$ in $[a, c)$, and thus $y(t)y''(t)>0$ in $[a, c)$. From this and $y''(c)>0$, it follows that $y''(t)>0$ and $y(t)>0$ in $[a, c)$. Since $y'(t)$ is an increasing function in $[a, c)$ and $y'(c)=0$, we have $y'(t)<0$, $t\in[a, c)$, which proves the lemma.

LEMMA 3.2. *If $p(t)\leqq 0$, $2q(t)-p'(t)\geqq d>0$ and $y(t)$ be a nontrivial solution of* (L) *for which $F(y(b))\geqq 0$ for some $b\in[a, \infty)$, then $y(t)$ is unbounded on $[a, \infty)$.*

PROOF. Suppose $y(t)$ is a nontrivial bounded solution of (L) with $F(y(b))\geqq 0$, $b\geqq a$. Since $p(t)$, $q(t)$ and $y(t)$ are bounded by Lemma 1.1, $y'(t)$ and $y''(t)$ are also bounded, and thus $F(y(t))$ is bounded. Hence

$$\int_a^\infty y^2(s)ds < \infty.$$

Since $y'(t)$ is bounded, it follows from the Lemmas 1.2 and 1.3 that $\lim_{t\to\infty} y^{(k)}(t)=0$, $k=0, 1, 2$. Thus $\lim_{t\to\infty} F(y,(t))=0$. This is a contradiction because $F(y(b))\geqq 0$ for $b\geqq a$ and $F(y(t))$ is an increasing function. Hence $y(t)$ is unbounded on $[a, \infty)$.

THEOREM 3.3. *If $p(t)\leqq 0$ and $2q(t)-p'(t)\geqq d>0$, then there exists a nontrivial solution $y(t)$ of* (L) *such that $y(t)>0$, $y'(t)<0$, $y''(t)>0$, for all $t\geqq a$ and $\lim_{t\to\infty} y(t)=\lim_{t\to\infty} y'(t)=0$. If $w(t)$ is any bounded solution of (L), then for some K, $w(t)=Ky(t)$.*

PROOF. Let $z_0(t)$, $z_1(t)$ and $z_2(t)$ be the three linearly independent solutions of (L). For each integer $n>a$ there exist numbers c_{0n}, c_{1n} and c_{2n} such that

$$\begin{aligned} &c_{0n}z_0(n) + c_{1n}z_1(n) + c_{2n}z_2(n) = 0, \\ (7) \qquad &c_{0n}z_0'(n) + c_{1n}z_1'(n) + c_{2n}z_2'(n) = 0, \\ &c_{0n}z_0''(n) + c_{1n}z_1''(n) + c_{2n}z_2''(n) > 0 \quad \text{and} \\ &c_{0n}^2 + c_{1n}^2 + c_{2n}^2 = 1. \end{aligned}$$

Let $y_n(t)$ be the solution of (L) defined by

$$y_n(t) = c_{0n}z_0(t) + c_{1n}z_1(t) + c_{2n}z_2(t).$$

By the independence of the solutions z_0, z_1 and z_2 and from (7), $y_n(t)$ is a nontrivial solution of (L) for which

$$y_n(n) = y_n'(n) = 0 \quad \text{and} \quad y_n''(n) > 0.$$

Thus by Lemma 3.1 it follows that

(8) $y_n(t) > 0, \quad y_n'(t) < 0 \quad \text{and} \quad y_n''(t) > 0 \quad \text{for all } t \in [a, n).$

By (7) there exists a sequence of integers $\{n_j\}$ and numbers c_i, $i=0, 1, 2$, such that $\lim_{n_j\to\infty} c_{in_j} = c_i$. Let $y(t)$ be the solution of (L) defined by

$$(9) \qquad y(t) = c_0z_0(t) + c_1z_1(t) + c_2z_2(t).$$

From the independence of z_0, z_1 and z_2 and

$$(10) \qquad c_0^2 + c_1^2 + c_2^2 = 1,$$

it follows that $y(t)$ is a nontrivial solution of (L). Since the sequences $\{y_{n_j}(t)\}$, $\{y_{n_j}'(t)\}$ and $\{y_{n_j}''(t)\}$ converge to the functions $y(t)$, $y'(t)$ and $y''(t)$ respectively on any finite subinterval of $[a, \infty)$, it follows from (8) that

(11) $y(t) \geqq 0, \quad y'(t) \leqq 0 \quad \text{and} \quad y''(t) \geqq 0 \quad \text{for } t \in [a, \infty).$

If equality held at a point $\bar{t}$ in the first inequality of (11), then $y(t) \equiv 0$ for $t \in [\bar{t}, \infty)$ which contradicts (9) and (10). Thus $y(t) > 0$, $t \in [a, \infty)$. Similarly, $y'(t) < 0$ and $y''(t) > 0$ for all $t \in [a, \infty)$. Since $y(t)$ is bounded by Lemma 3.2, $F[y(t)] < 0$ for $t > a$. Hence $\int_a^\infty y^2(t)dt < \infty$. Since $y'(t)$ is bounded, $y(t) \to 0$ as $t \to \infty$, and by Lemmas 1.1 and 1.3, $y^{(k)}(t) \to 0$ as $t \to \infty$, $k = 0, 1, 2, 3$.

To prove the last part of the theorem, let $w(t)$ be a bounded solution of (L), and let K be a number such that $w(b) - K\,y(b) = 0$. Consider the solution $Z(t) = w(t) - Ky(t)$. If $Z(t)$ were not identically equal to zero, then $F(Z(b)) \geqq 0$, and it would follow from Lemma 3.2 that $Z(t)$ could not be bounded, contradicting the boundedness of $w(t)$ and $y(t)$. This contradiction proves that $Z(t)$ is the triuial solution of (L) and hence $w(t) = Ky(t)$.

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