ON THE REINHARDT-MAHLER THEOREM

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1. Let $A_1, \cdots, A_n$ be n linearly independent points in R_n, the n-dimensional Euclidean space. The set $\Lambda = \{u_1A_1 + \cdots + u_nA_n: u_1, \cdots, u_n \text{ integers}\}$ is called a lattice, and $\{A_1, \cdots, A_n\}$ is called a base of Λ. Let A_i have co-ordinates $a_{1i}, \cdots, a_{ni}$. Then $d(\Lambda) = |\det(a_{ij})|$ is called the determinant of the lattice Λ; it is independent of the choice of a base of Λ.

Let S be a set in R_n. A lattice Λ is said to be *S-admissible* if Λ has no point other than the origin 0 in the interior of S. The *critical determinant* $\Delta(S)$ of S is defined by $\Delta(S) = \inf d(\Lambda)$, where Λ runs over all S-admissible lattices ($\Delta(S) = \infty$ if S has no admissible lattice). Clearly, $S \subset T$ implies $\Delta(S) \leqq \Delta(T)$.

One of the principal problems in Geometry of Numbers is to find a method for determining $\Delta(S)$ for a given set S. For two- and three-dimensional symmetrical[1] convex bodies, Minkowski reduced the problem to the discussion of special classes of lattices with points on the boundary of the body. This method has been partially extended to symmetrical convex bodies in R_4, but it is very difficult to apply in spaces of dimension higher than two. Reinhardt [7] and Mahler [4] independently proved that for a symmetrical convex domain K in R_2, $\Delta(K) = H(K)/4$, where $H(K)$ denotes the area of a smallest symmetrical "hexagon"[2] containing K. In other words, a symmetrical convex domain can be inscribed in a space-filling symmetrical convex domain with the same critical determinant. The straightforward generalization of this result to higher dimensions would be:

If K is a symmetrical convex body in R_n then K is contained in a space-filling symmetrical convex body $\mathcal{P}$ with $\Delta(K) = \Delta(\mathcal{P}) = V(\mathcal{P})/2^n$, where $V(\mathcal{P})$ denotes the volume of $\mathcal{P}$.

The object of this note is to prove that this generalization does not hold in R_n, for $n \geqq 3$.

2. Let $\mathcal{K}$ be the class of symmetrical open convex bodies in R_n.

DEFINITION. Let $K \in \mathcal{K}$. K is said to be *$\mathcal{K}$-maximal (for packings)* if $H \in \mathcal{K}$, $H \supsetneq K$ implies that $\Delta(H) > \Delta(K)$.

DEFINITION. A K-admissible lattice Λ with $d(\Lambda) = \Delta(K)$ is called a *critical lattice* of K.

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[1] By a symmetrical body we mean a body centered at 0, unless the contrary is evident in the context.

[2] By a "hexagon" we mean a polygon with at most six sides.

It is well known that if $K \in \mathcal{K}$, then K has at least one critical lattice. Let Λ be a critical lattice of K. One can easily construct (as in Davenport and van der Corput [1, pp. 410]) a polyhedron $\mathcal{P}$ containing K, such that Λ is $\mathcal{P}$-admissible, and so $\Delta(\mathcal{P}) = \Delta(K)$ $(d(\Lambda) = \Delta(K) \leqq \Delta(\mathcal{P}) \leqq d(\Lambda))$. This implies that $\mathcal{K}$-maximal bodies must be polyhedra. Now we prove

THEOREM 1. *Let $\mathcal{P}$ be a polyhedron. Then $\mathcal{P}$ is $\mathcal{K}$-maximal iff every critical lattice of $\mathcal{P}$ has at least one point in the $(n-1)$-dimensional interior of each $(n-1)$-dimensional face of $\mathcal{P}$.*

PROOF. Suppose $\mathcal{P}$ is $\mathcal{K}$-maximal and $\mathcal{P}$ has a critical lattice Λ with no point in the interior of an $(n-1)$-dimensional face F. Since Λ is a discrete set, moving the faces $\pm F$ parallel to themselves we can find a larger set $\mathcal{P}' \in \mathcal{K}$ for which Λ is admissible. Since $\Delta(\mathcal{P}') = \Delta(\mathcal{P})$, this gives a contradiction to the definition of maximality.

Next suppose that every critical lattice of $\mathcal{P}$ has a point in the interior of each $(n-1)$-dimensional face of $\mathcal{P}$. Let $S \in \mathcal{K}$, $S \not\supseteq \mathcal{P}$. Let $P \in S$, $P \notin \mathcal{P}$. Join OP. Let OP meet the boundary of $\mathcal{P}$ at P_1. Then $P_1 \in S$. Since S is open, there is a neighborhood of P_1 contained in S. Consequently, we can find a point P_2 in S which also lies in the $(n-1)$-dimensional interior of an $(n-1)$-dimensional face F of $\mathcal{P}$ containing P_1. Since S contains the convex cover of $\mathcal{P}$ and P, S also contains the interior of the face F. Since F contains in its interior a point of every critical lattice of $\mathcal{P}$, no critical lattice of $\mathcal{P}$ can be S-admissible. Hence, no lattice Λ with $d(\Lambda) = \Delta(\mathcal{P})$ is S-admissible. Since S possesses critical lattices, it follows that $\Delta(S) > \Delta(\mathcal{P})$. This proves that $\mathcal{P}$ is $\mathcal{K}$-maximal.

3. In this section we give an example of a polyhedron in R_3 which is $\mathcal{K}$-maximal but which is not space-filling.

Let $\mathcal{P}$ be the octahedron

$$|x| + |y| + |z| < 1.$$

Then $V(\mathcal{P})$ = volume of $\mathcal{P} = 4/3$, while Minkowski [5] proved that

$$\Delta(\mathcal{P}) = \frac{19}{108} \neq \frac{1}{8} V(\mathcal{P}).$$

Hence $\mathcal{P}$ is not space-filling.

Minkowski [5] also proved that the only critical lattice of $\mathcal{P}$ (up to automorphisms of K) is the lattice Λ generated by the points $(-1/3, 1/2, 1/6)$, $(1/6, -1/3, 1/2)$ and $(1/2, 1/6, -1/3)$. It is obvious that Λ and hence all other critical lattices of $\mathcal{P}$ have a point in the interior of each 2-dimensional face of $\mathcal{P}$. Therefore, by Theorem 1, $\mathcal{P}$ is $\mathcal{K}$-maximal.

REMARK. This example was suggested by Professor C. A. Rogers. Our original example was the cut cube

$$|x+y+z| < 1/2, \quad |x| < 1, \quad |y| < 1, \quad |z| < 1$$

whose critical lattices were determined by Whitworth [8].

4. We next prove

THEOREM 2. *For each $n>2$, there exist $\mathcal{K}$-maximal polyhedra in R_n which are not space-filling.*

As an immediate consequence we have

THEOREM 3. *The straightforward generalization of the Reinhardt-Mahler theorem (stated in the introduction) to R_n ($n \geqq 3$) is not true.*

We need two lemmas.

LEMMA 1. *Let $K \in \mathcal{K}$ be space-filling. Then every $(n-1)$-dimensional face of K is a finite union of nonoverlapping $(n-1)$-dimensional symmetric convex bodies.*

For $n=3$, this lemma was proved by Minkowski (see Hancock [2]). The same method applies in higher dimensions.

Let $V_n(K)$ denote the n-dimensional volume of a set K.

LEMMA 2. *A simplex can not be a finite union of nonoverlapping symmetrical convex bodies.*

PROOF (suggested by A. Heppes). Let S be a simplex in R_n. Let S be the convex cover of $(n+1)$ points $A_0, A_1, \cdots, A_n$. Since symmetrical convex bodies transform into symmetrical convex bodies under linear transformations ($Y = AX + B$, A nonsingular), we can suppose that $A_0 = O$, $A_1 = (1, 0, \cdots, 0), \cdots, A_n = (0, \cdots, 1)$.

Suppose S is the union of a finite number of symmetrical convex bodies $K_1, \cdots, K_m$. Let F be the face of S contained in $x_n = 0$, then

$$0 < V_{n-1}(F) = V_{n-1}(F \cap (\cup \overline{K}_i)) = \sum V_{n-1}(F \cap \overline{K}_i)$$

and $V_{n-1}(F \cap \overline{K}_i) > 0$ for at least one i, say $i=1$. So K_1 has a face F_4, of volume $V_{n-1}(F_1) = V_{n-1}(F \cap \overline{K}_1) > 0$, contained in $x_n = 0$. The parallel face F_2 lies in $x_n = h_1$, where $h_1 > 0$, and $V_{n-1}(F_2) = V_{n-1}(F_1) > 0$. Since $K_1 \subset S$, $h_1 < 1$. Hence there are K_i which lie above $x_n = h_1$ (in the obvious sense) and whose closures $\overline{K}_i$ have points in common with the face F_1. Let these be $K_2, \cdots, K_{m_1}$. Then

$$0 < V_{n-1}(F_2) = V_{n-1}\left(F_2 \cap \left(\bigcup_2^{m_1} \overline{K}_i\right)\right)$$
$$= \sum_{i=2}^{m_1} V_{n-1}(F_2 \cap \overline{K}_i),$$

so that there exists a set, say K_2, such that $V_{n-1}(F_2 \cap \overline{K}_2) > 0$. Therefore K_2 has a face F_3 in $x_n = h_1$ with $V_{n-1}(F_3) > 0$. The parallel face F_1 of K_2 with $V_{n-1}(F_4) = V_{n-1}(F_3) > 0$, is in $x_n = h_2$, where $0 < h_1 < h_2 < 1$ (since this face is contained in S). Repeating this argument, we would get a sequence $K_1, K_2, \cdots$ of sets K_i, and faces $F_2, F_4, \cdots$ lying in the planes $x_n = h_1, h_2, \cdots$, where $0 < h_1 < h_2 < \cdots < 1$. Since the K_i's are finitely many, this is impossible.

PROOF OF THEOREM 2. Let K be the cube $|x_i| < 1, 1 \leqq i \leqq n$ .Ollernshaw [6] proved that K is irreducible in $\mathcal{K}$ i.e. if $H \in \mathcal{K}$ and $H \subsetneqq K$ then $\Delta(H) < \Delta(K)$.

For $\epsilon > 0$, consider

$$K_\epsilon: |x_i| < 1, \quad 1 \leqq i \leqq n, \quad |x_1 + \cdots + x_n| < n - \epsilon.$$

Then $\Delta(K_\epsilon) < \Delta(K)$. Fix an $\epsilon < 2/(n+1)$. Define

$$\delta = \inf\{\epsilon' : \epsilon' \leqq \epsilon, \Delta(K_{\epsilon'}) = \Delta(K_\epsilon)\} \cdot$$

By a theorem of Mahler [3], it follows that

$$\Delta(K_\delta) = \Delta(K_\epsilon) < \Delta(K) = 1.$$

Define

$$K_{\delta,1}: \begin{cases} |x_i| < 1, & 2 \leqq i \leqq n, \\ |x_1 + \cdots + x_n| < n - \delta. \end{cases}$$

Then

$$V(K_{\delta,1}) = 2^n(n - \delta) > 2^n.$$

Hence

$$\Delta(K_{\delta,1}) > 1 > \Delta(K_\delta).$$

Therefore, by the same argument as above, there exists $k_1 \geqq 1$ such that the critical determinant of $K_{\delta,1} \cap \{|x_1| < k_1\}$ is equal to $\Delta(K_\delta)$ and k_1 is maximal with this property. Applying this argument successively to other coordinates we get a body

$$H_\delta: |x_i| < k_i, \quad 1 \leqq i \leqq n, \quad |x_1 + \cdots + x_n| < n - \delta,$$

where $k_i \geqq 1$, $\Delta(H_\delta)=\Delta(K_\delta)$, and each k_i is maximal with this property, i.e. if we expand H_δ by moving out any two parallel faces, the critical determinant is increased.

As in the proof of Theorem 1, one can show that every critical lattice of H_δ has a point in the interior of each $(n-1)$-dimensional face of H_δ. Hence H_δ is $\mathcal{K}$-maximal. Now we shall prove that H_δ is not space-filling. By Lemmas 1 and 2, it is enough to prove that one face of H_δ is a simplex.

It can be easily shown that

$$V(H_\delta) \geqq 2^n - 2\frac{\delta^n}{n} + 2\frac{k_{i-1}}{n}\left(2^{n-1} - \frac{\delta^{n-1}}{n-1}\right),$$

for each i.

Since $V(H_\delta) \leqq 2^n\Delta(H_\delta) < 2^n$, we must have $k_i < 1+\delta$, $1 \leqq i \leqq n$.

The n points $P_i = (y_1^{(i)}, \cdots, y_n^{(i)})$,
where

$$y_j^{(i)} = k_j \qquad \text{if } j \neq i,$$
$$y_i^{(i)} = (n-\delta) - \sum_{j\neq i;\, 1\leq j\leq n} k_j$$

are vertices of the plane face $x_1 + \cdots + x_n = n-\delta$. We assert that these are the only vertices. Any other possible vertex would be $Q=(z_1, \cdots, z_n)$, where for some fixed i,

$$z_i = (n-\delta) - \sum_{j\neq i} \delta_j k_j, \qquad z_j = \delta_j k_j \ (j \neq i),$$

$$\delta_j = \pm 1 \quad \text{and at least one } \delta_j \text{ is } -1.$$

Since

$$z_i = (n-\delta) - \sum_{j\neq i} \delta_j k_j$$
$$= (n-\delta) - \sum_{j\neq i} k_j + 2\sum{}' k_j$$

(where $\sum'$ is taken over those j for which $\delta_j = -1$)

$$\geqq (n-\delta) - (n-1)(1+\delta) + 2\sum{}' k_j$$
$$\geqq 3 - n\delta > 1+\delta \quad (\text{because } \delta < 2/(n+1))$$
$$> k_i$$

it follows that Q cannot be a vertex of H_δ, and our assertion is proved.

The points $P_1 - P_n, P_2 - P_n, \cdots, P_{n-1} - P_n$ have coordinates

$$(n - \delta - \sum k_i, 0, \cdots, 0, \sum k_i - (n - \delta))$$
$$(0, (n - \delta) - \sum k_i, 0, \cdots, 0, \sum k_i - (n - \delta)), \cdots,$$
$$(0, \cdots, 0, n - \delta - \sum k_i, \sum k_i - (n - \delta)),$$

and are easily seen to be linearly independent.

Therefore, the face of H_δ in $x_1 + \cdots + x_n = n - \delta$ is a simplex, and our proof is completed.

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