ON THE H_p-PROBLEM FOR FINITE p-GROUPS

GUY T. HOGAN AND WOLFGANG P. KAPPE

1. **Introduction.** In 1956 D. R. Hughes [2] conjectured that if G is any group, p any prime, and if $H_p(G)$ denotes the subgroup of G generated by the elements of order different from p, then $H_p(G)=1$, or $H_p(G)=G$, or $[G: H_p(G)]=p$. It has recently been shown that this conjecture is false. The counterexample constructed by G. E. Wall [6] is a finite 5-group G such that $[G: H_5(G)]=25$. However, for the following classes of groups the H_p-conjecture is known to be true:

regular p-groups,
$p=2$ (Hughes [2]),
$p=3$ (Straus-Szekeres [5]),
finite groups which are not p-groups (Hughes-Thompson [3]),
finite p-groups of class at most p (Zappa [7]).

In this paper we show that the H_p-conjecture is true for finite metabelian p-groups and finite p-groups with cyclic factors in the lower central series.

2. **Definitions and notation.**

$$[y, x] = y^{-1}x^{-1}yx, \qquad [y, {}_0x] = y, \qquad [y, {}_{i+1}x] = [[y, {}_ix], x],$$
$$[y, {}_ix, {}_jz] = [[y, {}_ix], {}_jz].$$

$[A, B]$ = subgroup generated by all $[a, b]$ with $a\in A$ and $b\in B$.
$G_2=[G, G]$, $G_{i+1}=[G_i, G]$,
G^p = subgroup generated by all g^p for $g\in G$,
$\phi(G)$ = Frattini subgroup of G.
If G is a finite p-group, then $\phi(G)=G_2G^p$.
$A-B$ = set of all $a\in A$ such that $a\notin B$.
$Z_1(G)$ = center of G, $Z_{i+1}(G)/Z_i(G)=Z_1(G/Z_i(G))$.
G has class $c(G)\leqq m$ if $G_{m+1}=1$. Then $G_i\subseteq Z_{m+1-i}(G)$.
G is an ECF-group if G/G_2 has exponent p and G_i/G_{i+1} is cyclic.
G is metabelian if $[G_2, G_2]=1$. For metabelian groups we write the group operation as addition.

The following identities will be used frequently:
(1) $[a, bc]=[a, c][a, b][a, b, c]$,
(2) $v^{-1}av=a[a, v]$.

3. Lemma 1 [1, Lemma 2.9, Lemma 3.3]. *Let K be an ECF-group of class $c(K)\geqq p+1$. Then K has subgroups $\gamma(K)$ and $\eta(K)$ with*

Received by the editors May 3, 1967 and, in revised form, December 11, 1967.

$[K:\gamma(K)]=[\gamma(K):\eta(K)]=p$ *such that*

$$K_i = \langle [w, {}_{i-1}s], K_{i+1}\rangle$$

and

$$w^p[w, {}_{p-1}s] \in K_{p+1}$$

for all $s \in K-\gamma(K)$ *and* $w \in \gamma(K)-\eta(K)$.

LEMMA 2. *An ECF-group of exponent p has class at most p.*

PROOF. Suppose $c(K) \geqq p+1$. Since $w^p=1$ Lemma 1 gives $[w, {}_{p-1}s] \in K_{p+1}$, hence $K_p=K_{p+1}$. But $c(K)\geqq p+1$ implies $K_p \neq K_{p+1}$, a contradiction.

LEMMA 3. *If G is metabelian and n an integer, then*

$$(3) \qquad [u, v, w] = [u, w, v] \qquad \text{for all } v, w \in G,\ u \in G_2,$$

$$(4) \qquad [v, nw] = \sum_{1\leqq i\leqq n} \binom{n}{i} [v, {}_iw],$$

$$(5) \qquad n(v-w) = nv + \left(\sum_{0<i+j<n} \binom{n}{i+j+1} [v, {}_iw, {}_jv] \right) - nw.$$

PROOF. Identities (3) and (4) follow immediately from (1) and (2). To prove (5) we observe $-nw+v-w=v+[v, nw]-(n+1)w$ and (2), so (5) follows from (3), (4) and induction on n.

LEMMA 4 [4, III. Satz]. *A finite metabelian p-group G of exponent p has class at most p. If G has two generators then $c(G)\leqq p-1$.*

4. THEOREM. *Let G be a finite p-group with $H_p(G)\neq 1$. Then $[G: H_p(G)]\leqq p$ if G satisfies one of the following conditions*:

(a) *G_i/G_{i+1} is cyclic for all $i \geqq 2$,*

(b) *G is metabelian.*

PROOF. Assume G is of minimal order such that $[G: H_p(G)]>p$, and let $N\neq 1$ be a minimal normal subgroup of G. Since G is a p-group, we have $N=\langle z\rangle$, $z^p=1$ and $z\in Z_1(G)$. There is some $y \in H_p(G)$ $\neq 1$ of order greater than p; hence $(yz)^p \neq 1$, $yz \in H_p(G)$ and $z=y^{-1}(yz)$ $\in H_p(G)$. Hence any minimal normal subgroup of G is contained in $H_p(G)$. Further G/N satisfies again conditions (a) or (b) because $G_iN/G_{i+1}N$ is a homomorphic image of G_i/G_{i+1}. Since $H_p(G/N)$ $\subseteq H_p(G)/N$, we have $[G/N: H_p(G/N)]\geqq [G: H_p(G)]>p$, and the minimality of G implies $H_p(G/N)=1$. Hence G/N has exponent p, so that $G^p\subseteq N$. Since $G^p=1$ implies $H_p(G)=1$, we have $G^p=N$ and N is the unique minimal normal subgroup of G.

(A) Assume G satisfies (a). Then G/N is an ECF-group since it has exponent p and satisfies (a). Lemma 2 implies $c(G/N) \leqq p$, hence $c(G) \leqq p+1$, and Zappa's result [7] gives $c(G) > p$. We havc $c(G) = p+1$, $G_{p+1} = N$ and G is also an ECF-group since N has order p. Suppose there is some $w \in \gamma(G) - \eta(G)$ such that $w \notin H_p(G)$. Then $w^p = 1$ by definition of $H_p(G)$, and Lemma 1 gives $[w, {}_{p-1}s] \in G_{p+1}$, hence $G_p = G_{p+1}$ in contradiction to $c(G) = p+1$. This proves $\gamma(G) - \eta(G) \subseteq H_p(G)$. Now, if $x \in \eta(G)$ and $y \in \gamma(G) - \eta(G) \subseteq H_p(G)$, then $yx \in \gamma(G) - \eta(G) \subseteq H_p(G)$, hence $x = y^{-1}(yx) \in H_p(G)$. This proves

$$\gamma(G) = (\gamma(G) - \eta(G)) \cup \eta(G) \subseteq H_p(G),$$

and $[G: H_p(G)] \leqq [G: \gamma(G)] = p$ contrary to the assumption $[G: H_p(G)] > p$.

(B) Assume G satisfies (b). Since G/N has exponent p, Lemma 4 gives $c(G/N) \leqq p$, $c(G) \leqq p+1$. There is a subgroup U of G with $[U: H_p(G)] = p^2$. The minimality of G and $H_p(G) = H_p(U)$ imply $G = U$. Hence $[G: H_p(G)] = p^2$, $G_2 \subseteq H_p(G)$ and $G = \langle a, b, H_p(G) \rangle$ with $a^p = b^p = 1$. Since $G^p = N$ and N is the unique minimal normal subgroup of G we see $G^p \subseteq G_2$, hence $\phi(G) = G_2 G^p = G_2$.

(i) $[G: \phi(G)] = p^3$.

Let $c \in H_p(G)$ be an element of order greater than p, and $W = \langle a, b, c \rangle$. Since $H_p(W) \subseteq W \cap H_p(G)$ and $G = WH_p(G)$ the isomorphism theorem gives

$$\begin{aligned} p^2 = [G: H_p(G)] &= [WH_p(G): H_p(G)] \\ &= [W: W \cap H_p(G)] \leqq [W: H_p(W)]. \end{aligned}$$

The minimality of G and $1 \neq c \in H_p(G)$ imply $G = W = \langle a, b, c \rangle$, hence $[G: \phi(G)] \leqq p^3$. Next assume $[G: \phi(G)] = p^2$. Then $G_2 = \phi(G) = H_p(G)$ and $c(G/N) \leqq p-1$ by Lemma 4, hence $c(G) \leqq p$. This contradicts $[G: H_p(G)] > p$ either by Zappa's result [7] or directly as follows.

Since $G^p = N \subseteq Z_1(G)$, we have $0 = [w, pu] = p[w, u]$ for $w \in G$ and $u \in G_2$. If $w \in G - H_p(G)$ and $v \in G_2 = H_p(G)$, then $v - w \in G - H_p(G)$. Hence $pw = 0 = p(v-w)$ and (4) gives $0 = pv + [v, {}_{p-1}w]$. But $c(G) \leqq p$ implies $G_2 \subseteq Z_{p-1}(G)$, and $c(\langle w, Z_{p-1}(G) \rangle) \leqq p-1$. In particular $[v, {}_{p-1}w] = 0$ and hence $pv = 0$ for all $v \in G_2 = H_p(G)$, a contradiction.

(ii) G_2 has exponent p, and all proper subgroups of G have class at most p.

Let $W = \langle a, b, G_2 \rangle$ and observe that $G = WH_p(G)$ and $H_p(W) \subseteq W \cap H_p(G)$. Hence

$$\begin{aligned} p^2 = [G: H_p(G)] &= [WH_p(G): H_p(G)] \\ &= [W: W \cap H_p(G)] \leqq [W: H_p(W)]. \end{aligned}$$

The minimality of G and $W \neq G$ imply $H_p(W)=1$ and thus W and G_2 have exponent p. Next let V be a maximal subgroup of G. Then $V=\langle u, v, \phi(G)\rangle$ and $\phi(G)=G_2 \subseteq Z_p(G)$ by $c(G) \leqq p+1$. But $c(\langle u, v, N\rangle/N) \leqq p-1$, by Lemma 4, hence $c(\langle u, v\rangle) \leqq p$, and $c(Z_p(G)\langle u, v\rangle) \leqq p$.

(iii) $0=pz+[z, {}_{p-1}x]$ for $z \in H_p(G)$ and $x \in G-H_p(G)$.

Since $z-x \notin H_p(G)$ it follows from (5) and (ii) that

$$0 = pz + \sum_{i+j=p-1} [z, {}_i x, {}_j z].$$

We substitute $nx \notin H_p(G)$ for x. Since $n^{p-1} \equiv 1 \bmod p$, and $c(\langle x, z\rangle) \leqq p$ by (i) and (ii), we get a system of $p-1$ equations for the quantities $(pz+[z, {}_{p-1}x])$ and $[z, {}_i x, {}_{p-1-i} z]$ for $i<p-1$:

$$0 = pz + \sum_{i+j=p-1} n^i [z, {}_i x, {}_{p-1-i} z], \qquad n = 1, \cdots, p-1.$$

The determinant $\det(n^i)$ is a Vandermonde determinant and $\det(n^i) \not\equiv 0 \bmod p$. Hence $0=pz+[z, {}_{p-1}x]$ and $0=[z, {}_i x, {}_{p-1-i} z]$ for $i<p-1$.

(iv) $0=[u, {}_i a, {}_j b]$ for $u \in G_2$ and $i+j \geqq p-1$.

We substitute $u \in G_2 \subseteq H_p(G)$ for z and $na+b \in G-H_p(G)$ for x in (iii). Since $c(G) \leqq p+1$ clearly $[u, {}_i a, {}_j b] \in Z_1(G)$ for $i+j \geqq p-1$, and $pu=0$ by (ii), hence using (1) and (3) we get from (iii)

$$0 = \sum_{0<i<p} n^i \binom{p-1}{i} [u, {}_i a, {}_j b], \quad i+j = p-1; \quad n = 1, \cdots, p-1.$$

The Vandermonde argument gives

$$\binom{p-1}{i} [u, {}_i a, {}_j b] = 0$$

for $i+j=p-1$, hence (iv), since

$$\binom{p-1}{i} \not\equiv 0 \qquad \text{mod } p.$$

(v) $0=pz$ for all $z \in H_p(G)$, hence $H_p(G)=1$, a contradiction.

We substitute $na+b$ for x in (iii). From (iv) with $u=[z, a]$ or $u=[z, b]$ we see that the terms $[z, {}_i a, {}_j b]$ and $[z, {}_i b, {}_j a]$ with $i+j \geqq p$ are trivial which occur if we expand $[z, {}_{p-1} na+b]$ according to (1). Hence (iii) gives

$$0 = pz + \sum_{0<i<p} n^i X_i, \qquad n = 1, \cdots, p-1,$$

where

$$X_i = \frac{1}{2}\binom{p-1}{i}([z, {}_ia, {}_{p-1-i}b] + [z, {}_{p-1-i}b, {}_ia]) \qquad \text{for } i < p-1,$$

and $X_{p-1} = [z, {}_{p-1}a] + [z, {}_{p-1}b]$. The Vandermonde argument gives in particular $0 = pz + X_{p-1}$. But $a, b \notin H_p(G)$, so (iii) shows $0 = pz + [z, {}_{p-1}a] = pz + [z, {}_{p-1}b]$, hence $pz = 0$.

EXAMPLE. For regular p-groups $H_p(G) \neq 1$ implies $H_p(G) = G$. The following example shows that conditions (a) and (b) together with $H_p(G) \neq 1$ do not imply $H_p(G) = G$. Let A be an abelian group with generators $x_1, \cdots, x_{p-1}$ and defining relations $p^2x_1 = 0 = px_2 = \cdots = px_{p-1}$. The mapping σ defined by $x_i^\sigma = x_i + x_{i+1}$ for $1 \leqq i \leqq p-2$ and $x_{p-1}^\sigma = x_{p-1} - px_1$ preserves these relations, hence σ is an automorphism of A. Let G be the splitting extension of A by $\langle\sigma\rangle$. It follows from (4) that σ has order p, and we have $G_i = \langle x_i, \cdots, x_{p-1}, px_1\rangle$ for $i \leqq p-1$ and $G_p = \langle px_1\rangle$. In particular $c(G) = p$, G is metabelian and G_i/G_{i+1} is cyclic for all i. To prove $[G: H_p(G)] = p$ it is sufficient to show $p(a - \sigma) = 0$ for all $a \in A$ since $A \subseteq H_p(G)$. If $a = \alpha_1x_1 + \cdots + \alpha_{p-1}x_{p-1}$, then (5) gives $p(a-\sigma) = pa + [a, {}_{p-1}\sigma] = p\alpha_1x_1 - p\alpha_1x_1 = 0$.

REFERENCES

1. N. Blackburn, *On a special class of p-groups*, Acta Math. **100** (1958), 45–92.

2. D. R. Hughes, *A problem in group theory*, Bull. Amer. Math. Soc. **63**. (1957), 209.

3. D. R. Hughes and J. Thompson, *The H_p-problem and the structure of H_p-groups*, Pacific J. Math. **9** (1959), 1097–1102.

4. H. Meier-Wunderli, *Metabelsche Gruppen*, Comment. Helv. **25** (1951), 1–10.

5. E. G. Straus and G. Szekeres, *On a problem of D. R. Hughes*. Proc. Amer. Math. Soc. **9** (1958), 157–158.

6. G. E. Wall, *On Hughes' H_p-problem*, Proc. Internat. Conf. Theory of Groups, Austral. Nat. Univ., Canberra (August 1965), Gordon and Breach, New York, 1967.

7. G. Zappa, *Contributo allo studio del problema di Hughes sui gruppi*, Ann. Mat. Pura Appl. **57** (1962), 211–219.

OHIO STATE UNIVERSITY