COMPLEX POLYNOMIAL INTERPOLATION TO CONTINUOUS BOUNDARY DATA

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1. **Introduction.** Let Ω be a bounded Jordan region in the complex z-plane whose boundary is denoted by Γ and let $\{S^{(n)}\}_{n=1}^{\infty}$ be a family of finite sequences of points on Γ. J. H. Curtiss [2] has posed the problem of giving conditions on the curve Γ and on the families $\{S^{(n)}\}_{n=1}^{\infty}$ so that, for any complex valued continuous function f defined on Γ, the sequence of polynomials $\{P_n\}_{n=1}^{\infty}$, where P_n is a polynomial of appropriate degree which takes the same values as f in $S^{(n)}$, converges in Ω. In [2] Curtiss obtains convergence theorems for interpolation in the Fejér points for suitably smooth curves Γ. In this note we make use of these results and we obtain similar convergence theorems for a wider class of interpolation points. The conditions on the families given here are sufficient and examples are provided to show that, *in the form stated*, they cannot be weakened.

2. **Asymptotically neutral families and a theorem of Curtiss.** For technical convenience we suppose that the origin of the z-plane is contained in Ω. Let $z=\Phi(w)$ be the one-to-one conformal mapping of the exterior of the unit disc in the w-plane onto the complement of $\bar{\Omega}$, normalized by requiring that $\Phi(\infty)=\infty$ and $\Phi'(\infty)>0$. There is then a natural parametrization ϕ of Γ obtained by extending Φ to a homeomorphism of $|w|\geqq 1$ onto the complement of Ω and setting $\phi(t)=\Phi(e^{2\pi i t})$, $0\leqq t\leqq 1$. Extend the function ϕ for all real values of t by periodicity.

Let $S^{(n)}=\{z_k^{(n)}\}_{k=1}^{n}$, $n=1, 2, \cdots$, be a family of sets of points on Γ. We suppose that for each n the set $S^{(n)}$ consists of n distinct points and that they are numbered in such a way that $z_k^{(n)}=\phi(t_k^{(n)})$ for $k=1, 2, \cdots, n$, where $0\leqq t_1^{(n)}<t_2^{(n)}<\cdots<t_n^{(n)}<1$. The family $\{T^{(n)}\}$, $T^{(n)}=\{t_k^{(n)}\}_{k=1}^{n}$ for $n=1, 2, \cdots$, will be referred to as the *parameter family* associated with the family $\{S^{(n)}\}$.

We form the Lagrange polynomials $L_n[f;\cdot]$ which interpolate in the set $S^{(n)}$ to a function f defined on Γ. Introducing $\omega_n(z) =\prod_{k=1}^{n}(z-z_k^{(n)})$, we have

$$L_n[f; z] = \sum_{k=1}^{n} \frac{f(z_k^{(n)})\omega_n(z)}{\omega_n'(z_k^{(n)})(z-z_k^{(n)})}.$$

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The families $\{S^{(n)}\}$ most frequently occurring in the literature are the Fejér points, that is, the family whose associated parameter family is given by $t_k^{(n)}=(k-1)/n$, $k=1, 2, \cdots, n$, $n=1, 2, \cdots$. For such families on rectifiable curves we have [1] and [6]

$$(1) \qquad \lim_{n\to\infty} -\omega_n(z)/c^n = 1, \qquad z\in\Omega,$$

where c is the logarithmic capacity of Γ and the convergence is uniform on any closed subset of Ω. Families for which (1) holds have been called *equilibrium* (in [6]) and, more recently, *asymptotically neutral* (in [4]), and are known to be of importance in many problems in approximation theory. In particular, asymptotically neutral families are appropriate for the problem considered here. Results obtained by Curtiss in [2], when stated in our terminology, provide the following basic theorem.

THEOREM 1 (CURTISS). *Let Ω be a bounded Jordan region containing the origin whose boundary Γ is rectifiable and let $\{S^{(n)}\}_{n=1}^{\infty}$ be an asymptotically neutral family for which the Lagrange interpolation operators $\{L_n[\cdot;\, z]\}_{n=1}^{\infty}$ are uniformly bounded for z in any closed subset of Ω. Then for every complex valued continuous function f on Γ the Lagrange interpolation polynomials $\{L_n[f;\, z]\}_{n=1}^{\infty}$ converge to the Cauchy integral of f at z and uniformly on closed subsets of Ω.*

3. Results. It is the purpose of this note to point out that there is a simple condition on the parameter family $\{T^{(n)}\}$ which insures both that $\{S^{(n)}\}$ is asymptotically neutral and that the Lagrange interpolation operators are uniformly bounded on closed subsets. If we set $t_{n+1}^{(n)}=1+t_1^{(n)}$ for $n=1, 2, \cdots$, then this condition is

$$(*) \qquad \sum_{k=1}^{n}\left| t_{k+1}^{(n)} - t_k^{(n)} - \frac{1}{n}\right| = o(n^{-1}), \qquad \text{as } n\to\infty.$$

THEOREM 2 [7, THEOREM 4]. *Let Γ be a rectifiable Jordan curve. If the parameter family $\{T^{(n)}\}$ associated with the family $\{S^{(n)}\}$ satisfies* (*), *then the family $\{S^{(n)}\}$ is asymptotically neutral on Γ.*

The condition (*) is sharp for this theorem in the sense that if the $o(n^{-1})$ on the right-hand side is replaced by $O(n^{-1})$ then in general the result is false. There are examples in [7].

In order to obtain boundedness for the Lagrange interpolation operators it is convenient to restrict somewhat the class of Jordan regions considered. We will say that the Jordan region Ω has a *smooth* boundary if the mapping function Φ has a nonzero derivative on

$|w| = 1$ which satisfies a Lipschitz condition there. With this restriction we can prove

THEOREM 3. *Let Ω be a bounded Jordan region with a smooth boundary and let the family $\{S^{(n)}\}$ be such that the associated parameter family satisfies* (*). *Then the Lagrange interpolation operators are uniformly bounded on closed subsets of Ω.*

The proof of this theorem is somewhat technical and is presented as two lemmas.

We begin by observing that

$$\|L_n[\cdot\,;z]\| = \sum_{k=1}^{n} |\omega_n(z)|\,|\omega_n'(z_k^{(n)})(z - z_k^{(n)})|^{-1}.$$

On any closed subset of Ω, the factor $|z - z_k^{(n)}|$ is bounded uniformly away from zero, and, by Theorem 2, we have $|\omega_n(z)/c^n| \to 1$ uniformly. Thus it is sufficient to consider

$$\sum_{k=1}^{n} \left| \frac{c^n}{\omega_n'(z_k^{(n)})} \right| = \sum_{k=1}^{n} c \left\{ \prod_{j=1;\, j\neq k}^{n} c/|z_k^{(n)} - z_j^{(n)}| \right\}$$

$$= c \sum_{k=1}^{n} \left\{ \prod_{j=1;\, j\neq k}^{n} |w_k^{(n)} - w_j^{(n)}| \right\}^{-1} \left\{ \prod_{j=1;\, j\neq k}^{n} \left| \frac{c(w_k^{(n)} - w_j^{(n)})}{\Phi(w_k^{(n)}) - \Phi(w_j^{(n)})} \right| \right\},$$

where we have written $z_k^{(n)} = \Phi(w_k^{(n)})$, $k = 1, 2, \cdots, n$, $n = 1, 2, \cdots$.

LEMMA 1. *If the parameter family $\{T^{(n)}\}$ satisfies* (*), *then*

$$\limsup_{n\to\infty} \sum_{k=1}^{n} \prod_{j=1;\, j\neq k}^{n} |w_k^{(n)} - w_j^{(n)}|^{-1} \leq 1.$$

PROOF. For notational convenience we will no longer explicitly indicate the dependence of $z_k^{(n)}$, $w_k^{(n)}$, and $t_k^{(n)}$ on n. Define $t_{n+k} = 1 + t_k$ for $1 \leq k \leq n$. Then using the fact that $\prod_{k=1}^{n-1} |1 - \exp(2\pi i k/n)| = n$ and (*), we have for sufficiently large integers n,

$$I(n) = \sum_{k=1}^{n} \prod_{j=1;\, j\neq k}^{n} |w_k - w_j|^{-1}$$

$$= \frac{1}{n} \sum_{k=1}^{n} \prod_{j=1;\, j\neq k}^{n} \left| \frac{1 - \exp(2\pi i j/n)}{1 - \exp[2\pi i(t_{k+j} - t_k)]} \right|$$

$$\leq \frac{1}{n} \sum_{k=1}^{n} \prod_{j=1;\, j\neq k}^{n} \left\{ 1 + \frac{n}{j} \left| t_{k+j} - t_k - \frac{j}{n} \right| \right\}.$$

Next, set $\epsilon_k = n|t_{k+1} - t_k - 1/n|$, $k = 1, 2, \cdots, 2n-1$, and note that since $\epsilon_{n+k} = \epsilon_k$ for $k = 1, 2, \cdots, n-1$, the quantity $\sum_{j=1}^{n} \epsilon_{k+j-1}$ is independent of k for $1 \leqq k \leqq n$, and its value, call it ϵ, approaches zero as n tends to infinity by (*). We suppose n to be taken so large that $0 \leqq \epsilon < 1$ and since the lemma is trivial if $\epsilon = 0$, we assume $\epsilon > 0$. Then $n|t_{k+j} - t_k - j/n| \leqq \sum_{m=1}^{j} \epsilon_{k+m-1}$, and thus,

$$\begin{aligned} I(n) &\leqq \frac{1}{n} \sum_{k=1}^{n} \exp \left\{ \sum_{j=1}^{n-1} \frac{1}{j} \left[\sum_{m=1}^{j} \epsilon_{k+m-1} \right] \right\} \\ &\leqq \frac{1}{n} \sum_{k=1}^{n} \exp \left\{ \sum_{m=1}^{n-1} \epsilon_{k+m-1} \left[\sum_{j=m}^{n-1} \frac{1}{j} \right] \right\} \\ &\leqq n^{\epsilon-1} \exp \epsilon \sum_{k=1}^{n} (1)^{\epsilon_k} (1)^{\epsilon_{k+1}} \left(\frac{1}{2} \right)^{\epsilon_{k+2}} \cdots \left(\frac{1}{n-1} \right)^{\epsilon_{k+n-1}}. \end{aligned}$$

Applying a version of Hölder's inequality, we obtain

$$I(n) \leqq n^{\epsilon-1} \exp \epsilon \prod_{k=1}^{n} \left[1 + \sum_{m=1}^{n-1} m^{-\epsilon} \right]^{\epsilon_k/\epsilon} \leqq \frac{\exp \epsilon\,(1 + n^{\epsilon-1})}{1-\epsilon},$$

which completes the proof. (The author thanks the referee for pointing out an error in the proof of a somewhat stronger version of this lemma.)

LEMMA 2. *If Ω is a bounded Jordan region with a smooth boundary and if the parameter family associated with $\{S^{(n)}\}$ satisfies (*), then there is a constant M such that*

$$\prod_{j=1;\, j\neq k}^{n} \left| \frac{c(w_j^{(n)} - w_k^{(n)})}{\Phi(w_j^{(n)}) - \Phi(w_k^{(n)})} \right| \leqq M, \qquad k = 1, \cdots, n, \quad n = 1, 2, \cdots.$$

PROOF. The technique used in the proof of this lemma is a familiar one. After taking the logarithm of the product, we estimate the resulting sum by interpreting it as a Riemann sum and comparing it with an appropriate integral.

For each integer k, $k = 1, 2, \cdots, n$, we define

$$\begin{aligned} \Psi_k(w) &= \frac{\Phi(w) - \Phi(w_k)}{c(w - w_k)}, \qquad & w \neq w_k, \quad |w| \geqq 1, \\ &= \Phi'(w_k)/c, & w = w_k, \end{aligned}$$

and observe that Ψ_k is holomorphic in $|w| > 1$ and continuous in $|w| \geqq 1$. By the Cauchy integral theorem (cf. [1] and [5, Lemma 2.2]) we have

$$\int_0^1 \log \Psi_k(e^{2\pi i t})dt = 0. \tag{2}$$

Since Φ' is Lipschitzian on $|w| = 1$, it follows that $\psi_k(t) = \Psi_k(e^{2\pi i t})$ is absolutely continuous on $0 \leqq t \leqq 1$. By a result of [7, Equations (6) ff.], we have for any family $\{T^{(n)}\}$ satisfying (*) the inequality

$$\left| \sum_{j=1;\, j\neq k}^{n} \log|\psi_k(t_j)| + \log|\Phi'(w_k)/c| - n\int_0^1 \log|\psi_k(t)|\,dt \right|$$

$$\leqq M \int_0^1 \left| \frac{d}{dt} \log|\psi_k(t)| \right| dt,$$

where M is a constant independent of k and n. Therefore, making use of (2) and the hypothesis that $|\Phi'|$ is bounded above and away from zero, the proof of the lemma will be completed by showing that the quantity

$$B_k = \int_0^1 \left| \frac{d}{dt} \log|\psi_k(t)| \right| dt$$

is uniformly bounded for $k = 1, \cdots, n$, and all n. However, this follows from

$$B_k \leqq \left\{ \min_{|w|=1} |\Phi'(w)| \right\}^{-1} \int_0^1 \left| \frac{d}{dt} \psi_k(t) \right| dt$$

$$\leqq \frac{\pi^3 K}{2c} \left\{ \min_{|w|=1} |\Phi'(w)| \right\}^{-1},$$

where K is the Lipschitz constant for Φ'.

We remark that this proof also works if the condition that Φ' be Lipschitzian is replaced by the condition that $|\Phi''(w)| \log^+ |\Phi''(w)|$ is integrable on $|w| = 1$. In this case, the uniform estimate on the B_k is obtained by appealing to a theorem of Hardy and Littlewood [**3**, p. 102].

4. Examples. The condition on the family $\{S^{(n)}\}$ imposed here is the weakest possible in the sense that if the $o(n^{-1})$ on the right-hand side of (*) is replaced by $O(n^{-1})$, then the Lagrange interpolation polynomials need not converge to the Cauchy integral of the function. As an example, take Γ to be the unit circle, $S^{(n)}$ the set of n points consisting of the $(n+1)$th roots of unity with $z = 1$ removed, and $f(z) = 1/z$. Then with the notation introduced above,

$$\sum_{k=1}^{n}\left|t_{k+1}^{(n)}-t_k^{(n)}-\frac{1}{n}\right|=\left\{\sum_{k=1}^{n-1}\left|\frac{1}{n+1}-\frac{1}{n}\right|\right\}+\left|\frac{2}{n+1}-\frac{1}{n}\right|$$
$$\leqq \frac{3}{n+1}.$$

However,

$$L_n[f;z]=L_n[1/z;z]=\frac{1}{z}\left\{1-\frac{\omega_n(z)}{\omega_n(0)}\right\},$$

which converges to $1/(z-1)$ as n increases to infinity, whereas the Cauchy integral of f is zero.

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