ON THE SUMMABILITY OF THE DIFFERENTIATED FOURIER SERIES. II

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In the previous paper [1] we established the following generalization of Fatou's theorem on the Abel summability of the differentiated Fourier series.

Let $f \in L(0, 2\pi)$ *of period* 2π *with* $\psi_{x_0}(t) = f(x_0+t) - f(x_0-t)$ *essentially bounded*[2] *in a neighborhood of* $t=0$. *Then if, for* $\alpha = 2$, $\alpha - f'_{aps}(x_0) = y$, *the differentiated Fourier series of* f *is Abel summable to* y *at* x_0.

We note that $y = \alpha - f'_{aps}(x_0)$, i.e. y is the *α-approximate symmetric derivative* of f at x_0, means that if, for any $\epsilon > 0$, $H_\epsilon = \{t: |y - \psi_{x_0}(t)/2t| \geqq \epsilon\}$, then $m(H_\epsilon \cap (-t, t)) = o(t^\alpha)$ as $t \to 0$.

We showed there that $\alpha = 2$ cannot be replaced by a smaller value and that essentially bounded cannot be omitted.

Here we consider the question of replacing essentially bounded by a weaker condition. Clearly the differentiability condition plus the essential boundedness implies the condition

$$(*) \qquad \int_0^t |\psi_{x_0}(u)|\, du = O(t^2)$$

as $t \to 0$, but as we shall see, this will not replace the essential boundedness.

We will say that f *satisfies condition* A_q at x_0 if for some sufficiently large M

$$\int_{E_M \cap (0,t)} |\psi_{x_0}(u)|\, du = o(t^q)$$

where $E_M = \{t: |\psi_{x_0}(t)| \geqq M\}$. Clearly A_q implies $A_{q'}$ for $q' < q$.

Our result is:

The requirement of essential boundedness in the generalized Fatou theorem can be replaced by condition A_q *with* $q = 2$. *This is best possible in the sense that* $o(t^2)$ *cannot be replaced by* $O(t^2)$.

Throughout this paper, we suppose, as in [1], that $x_0 = 0$, $f(0) = 0$, $y = 0$, and allow C to denote a positive constant not necessarily the same at each occurrence.

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[2] In the result of [1] we assumed the essential boundedness of f in a neighborhood of x_0, but this somewhat stronger result was actually proved there.

We begin by exhibiting the counterexample required to show that our result is best possible. Choose $\beta > 2$ and let $I_k = (1/2^k, 1/2^k + 1/2^{\beta k})$ and $E = \bigcup_{k=1}^{\infty} I_k$. Denoting the characteristic function of I_k by χ_{I_k}, we let

$$f = \sum_1^{\infty} 2^{(\beta-2)k} \chi_{I_k}$$

in $(0, 2\pi)$ and have period 2π. Then $f \in L(0, 2\pi)$ since

$$\int_0^{2\pi} |f(t)|\, dt = \sum_1^{\infty} 2^{-2k} < \infty.$$

For small t, $f(t) - f(-t) = 0$ except for $t \in E$, and if $1/2^n < t \leqq 1/2^{n-1}$, then

$$m(E \cap (0, t)) \leqq \sum_n^{\infty} 2^{-\beta k} = o(t^2) \qquad \text{as } t \to 0,$$

which shows that $\alpha - f'_{aps}(0) = 0$ for $\alpha = 2$. Also

$$t^{-2} \int_0^t |f(u) - f(-u)|\, du \leqq 2^{2n} \sum_{n-1}^{\infty} 2^{-2k} = C < \infty.$$

In a similar fashion we may show that condition A_2 is not satisfied. We note that f satisfies (*).

Let $P(r, t)$ denote the Poisson kernel. Applying the estimate

$$-\int_a^b P_t(r, t) dt > C\eta r(a + b)(b - a)/(\eta^4 + b^4),$$

where $0 < a < b < \pi/2$ and $\eta = 1 - r$, we have, choosing $\eta = 2^{-k}$,

$$u_t(r, 0) > -(1/\pi) \int_{I_k} 2^{(\beta-2)k} P_t(r, t) dt > C > 0$$

for all k, which establishes the negative part of our result.

We now suppose that $\psi(t) = \psi_{x_0}(t)$ satisfies A_2.

Write $f_1 = f\chi_{E_M}$, $f = f_1 + f_2$, and let ψ_1 and ψ_2 denote the corresponding symmetric differences. Thus $\psi = \psi_1 + \psi_2$. Denoting the Abel means of the Fourier series of f, f_1, and f_2 by u, u_1, and u_2 respectively, we have $u_t = (u_1)_t + (u_2)_t$. Since ψ_2 is bounded and $\alpha(f_2)'_{aps} = 0$ for $\alpha = 2$, we have $\lim_{r \to 1} (u_2(r, 0))_t = 0$. We see then that we may assume from the onset that $f = f_1$.

A_2 implies that for a given $\epsilon > 0$ there is a $\delta > 0$ such that

$$\int_0^t |\psi(u)|\,du < \epsilon t^2$$

for $t \in (0, \delta)$. We have

$$|u_t(r, 0)| < C\int_0^\pi |\psi(t)P_t(r, t)|\,dt = C\left(\int_0^\delta + \int_\delta^\pi\right) = \mathcal{J}_1 + \mathcal{J}_2.$$

Let $t_k = \delta 2^{-k}$ and $I_k = (t_{k+1}, t_k)$. Since $|P_t(r, t)| < C\eta t/(\eta^4 + t^4)$, we have

$$\mathcal{J}_1 \leqq C\sum_{k=0}^{\infty}\int_{I_k} |\psi(t)P_t(r, t)|\,dt \leqq C\eta \sum_k (t_k/\eta^4 + t_{k+1}^4))\int_{I_k} |\psi(t)|\,dt$$

$$< C\epsilon\eta \sum_k t_k^3/(\eta^4 + t_{k+1}^4).$$

It is easily verified that

$$2^7 \int_{I_k} t^2/(\eta^4 + t^4)dt > t_k^3/(\eta^4 + t_{k+1}^4).$$

Thus

$$\mathcal{J}_1 < C\epsilon\eta \int_0^\infty t^2/(\eta^4 + t^4)dt = C\epsilon.$$

We also have

$$\mathcal{J}_2 \leqq C\eta \int_\delta^\pi |\psi(t)|\, t/(\eta + t^4)dt < C\eta\delta^{-3}.$$

Finally,

$$|u_t(r, 0)| < C(\epsilon + \eta\delta^{-3}) < C\epsilon$$

if η is sufficiently small, the constant being independent of the choice of ϵ.

References

1. D. Waterman, *On the summability of the differentiated Fourier series*, Bull. Amer. Math. Soc. **73** (1967), 109–112.

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