REMARKS ON THE REDUCTION THEORY OF VON NEUMANN ALGEBRAS

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In [7, Problem 4, p. 3.55], S. Sakai proposed the following question: If any component of a measurable field $\gamma \mapsto \mathfrak{M}(\gamma)$ of von Neumann algebras over (Γ, μ) is isomorphic to some fixed von Neumann algebra $\mathfrak{M}_0$ with separable predual, where Γ means a locally compact Hausdorff space and μ a positive Radon measure on Γ, *is it true that the direct integral* $\int_\Gamma^\oplus \mathfrak{M}(\gamma) d\mu(\gamma)$ *is isomorphic to the tensor product* $\mathfrak{A} \otimes \mathfrak{M}_0$ *of the associated diagonal algebra* $\mathfrak{A} = L^\infty(\Gamma, \mu)$ *and* $\mathfrak{M}_0$? If Γ satisfies the second countability axiom, then one can easily settle the above problem affirmatively as an application of [1, Proposition 4, p. 187]. However, in the case that the countability assumption of Γ is dropped, J. Dixmier proposed the similar problem in [1, p. 175]. Namely, roughly speaking, if $\gamma \mapsto y_i(\gamma)$ is a measurable operator field over (Γ, μ) for each $i \in I$ and there exists a family of bounded operators $\{x_i\}_{i \in I}$ on a Hilbert space and an isometric operator $u(\gamma)$ for each γ such that $u(\gamma) x_i u(\gamma)^{-1} = y_i(\gamma)$ for every $\gamma \in \Gamma$ and $i \in I$, *does there exist an isometry which transforms* $1 \otimes x_i$ *to* $\int_\Gamma^\oplus y_i(\gamma) d\mu(\gamma)$ *for every* $i \in I$?

In this paper, we shall settle the above two related questions affirmatively.

Let Γ be a locally compact Hausdorff space and μ a positive Radon measure on Γ. Let $\gamma \mapsto H(\gamma)$ be a measurable field of Hilbert spaces over Γ in the sense of [1]. Let H_0 be a separable Hilbert space. If $H(\gamma) = H_0$ for every $\gamma \in \Gamma$ and the measurable vector fields are measurable H_0-valued functions over Γ, then $\gamma \mapsto H(\gamma)$ is called a constant field corresponding to H_0. If $\gamma \mapsto \dim_{\cdot} H(\gamma)$ is constant, then the measurable field $\gamma \mapsto H(\gamma)$ of Hilbert spaces is isomorphic to a constant measurable field corresponding to some Hilbert space H_0. Namely, there exist a Hilbert space H_0 and isometry $u(\gamma)$ of $H(\gamma)$ onto H_0 for every $\gamma \in \Gamma$ such that $\gamma \mapsto \xi(\gamma) \in H(\gamma)$ is a measurable vector field if and only if $u(\gamma)\xi(\gamma) \in H_0$ is a measurable H_0-valued function over Γ. Putting $(u\xi)(\gamma) = u(\gamma)\xi(\gamma)$ for every $\xi = \int_\Gamma^\oplus \xi(\gamma) d\mu(\gamma) \in H = \int_\Gamma^\oplus H(\gamma) d\mu(\gamma)$, u is an isometry of H onto the Hilbert space $L^2_{H_0}(\Gamma, \mu)$ of all square integrable H_0-valued functions over Γ. Therefore, H is isomorphic to the tensor product $L^2(\Gamma, \mu) \otimes H_0$.

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THEOREM 1. *Let H_0 be a separable Hilbert space and $\{x_i\}_{i\in I}$ a family of bounded operators on H_0. Let $\gamma\mapsto H(\gamma)$ be a measurable field of Hilbert spaces over Γ,*

$$H = \int_\Gamma^\oplus H(\gamma)d\mu(\gamma),$$

and, for each $i\in I$, $\gamma\mapsto y_i(\gamma)$ is a measurable essentially bounded operator field over Γ with the direct integral

$$y_i = \int_\Gamma^\oplus y_i(\gamma)d\mu(\gamma) \in \mathcal{B}(H).$$

Suppose that there exists, for every $\gamma\in\Gamma$, an isometry $u(\gamma)$ of H_0 onto $H(\gamma)$ such that

$$y_i(\gamma) = u(\gamma)x_iu(\gamma)^{-1}, \qquad i \in I.$$

Then there exists an isometry of $L^2(\Gamma, \mu)\otimes H_0$ onto H which transforms, for every $i\in I$, $1\otimes x_i$ into y_i.

PROOF. By the above consideration, we may assume that $\gamma\mapsto H(\gamma)$ is a constant field corresponding to H_0. Then $\gamma\mapsto y_i(\gamma)$ is a $B(H_0)$-valued measurable function over Γ, where the strong operator topology is considered in the full operator algebra $\mathcal{B}(H_0)$, so that $\gamma\mapsto u(\gamma)x_iu(\gamma)^{-1}$ is measurable for every $i\in I$. Let $\mathfrak{M}_0$ be the von Neumann algebra generated by $\{x_i\}_{i\in I}$. Let $\mathfrak{N}$ be the set of all bounded operators x on H_0 with the property; $\gamma\mapsto u(\gamma)xu(\gamma)^{-1}$ is measurable. Then $\mathfrak{N}$ is a *-algebra containing $\{x_i\}_{i\in I}$. If $\{a_n\}$ is a bounded sequence in $\mathfrak{N}$ strongly converging to $a_0\in\mathcal{B}(H_0)$, then $u(\gamma)a_nu(\gamma)^{-1}$ also converges strongly to $u(\gamma)a_0u(\gamma)^{-1}$, so that $\gamma\mapsto u(\gamma)a_0u(\gamma)^{-1}$ is measurable. By the separability of H_0, the strong operator topology in bounded parts of $\mathcal{B}(H_0)$ is metrizable, so that $\mathfrak{N}$ is strongly closed. Therefore, $\mathfrak{N}$ is a von Neumann algebra containing $\{x_i\}_{i\in I}$, and $\mathfrak{N}$ contains $\mathfrak{M}_0$. Thus, for every $x\in\mathfrak{M}_0$, $\gamma\mapsto u(\gamma)xu(\gamma)^{-1}$ is measurable. Again by the separability of H_0, there exists a separable strongly dense C^*-subalgebra A of $\mathfrak{M}_0$. Let $\mathrm{Rep}(A:H_0)$ denote the space of all representations of A on H_0. Then $\mathrm{Rep}(A:H_0)$ is a Polish space with respect to the simple convergence topology, where the strong operator topology is considered in $\mathcal{B}(H_0)$. Let $\mathfrak{U}$ denote the group of all unitary operators on H_0, which is a Polish group with respect to the strong operator topology. For each $u\in\mathfrak{U}$, define the action of u on $\pi\in\mathrm{Rep}(A:H_0)$ by $(u\pi)(x)=u\pi(x)u^{-1}$ for every $x\in A$. Then $\mathfrak{U}$ becomes a topological transformation group of $\mathrm{Rep}(A:H_0)$. Let π_0 be the identity representation of A on H_0. The stability group

$\mathfrak{U}_0$ of $\mathfrak{U}$ at π_0 is the unitary group of the commutant $\mathfrak{M}_0'$ of $\mathfrak{M}_0$, so that $\mathfrak{U}_0$ is a closed subgroup. Therefore, the quotient map: $\mathfrak{U}\mapsto\mathfrak{U}/\mathfrak{U}_0$ has a Borel transversal $\mathfrak{W}$ by [3, Lemma 3]. Putting $\Psi(u)=u\pi_0$ for $u\in\mathfrak{U}$, Ψ is a one-to-one Borel map of the standard Borel space $\mathfrak{W}$ into the standard Borel space $\operatorname{Rep}(A:H_0)$, so that it becomes a Borel isomorphism by [6, Theorem 3.2]. Let Φ denote the inverse map of Ψ defined on $\Psi(\mathfrak{W})$. By the measurability of the map: $\gamma\mapsto u(\gamma)xu(\gamma)^{-1}$, $\gamma\mapsto u(\gamma)\pi_0$ is a $\operatorname{Rep}(A:H_0)$-valued measurable function over Γ, so that $\gamma\mapsto\Phi(u(\gamma)\pi_0)=v(\gamma)$ is also a measurable $\mathfrak{U}$-valued function on Γ. By the equality

$$v(\gamma)\pi_0 = \Psi(v(\gamma)) = \Psi(u(\gamma)) \equiv u(\gamma)\pi_0,$$

$u(\gamma)^{-1}v(\gamma)$ belongs to $\mathfrak{U}_0$, so that $v(\gamma)xv(\gamma)^{-1}=u(\gamma)xu(\gamma)^{-1}$ for every $x\in\mathfrak{M}_0$, which implies that $y_i(\gamma)=v(\gamma)x_iv(\gamma)^{-1}$ for every γ of Γ and $i\in I$. Putting

$$v = \int_\Gamma^\oplus v(\gamma)d\mu(\gamma),$$

v becomes the desired isometry of $L^2(\Gamma,\mu)\otimes H_0$ onto H. This completes the proof.

Theorem 1 gives an affirmative answer to the question proposed by J. Dixmier in [1, p. 175].

THEOREM 2. *Let $\mathfrak{M}_0$ be a von Neumann algebra acting on a separable Hilbert space H_0. Let $\gamma\to H(\gamma)$ be a measurable field of Hilbert spaces over Γ,*

$$H = \int_\Gamma^\oplus H(\gamma)d\mu(\gamma),$$

$\mathfrak{A}$ the algebra of all diagonal operators, $\gamma\mapsto\mathfrak{M}(\gamma)\subset\mathfrak{B}(H(\gamma))$ a measurable field of von Neumann algebras and

$$\mathfrak{M} = \int_\Gamma^\oplus \mathfrak{M}(\gamma)d\mu(\gamma).$$

Suppose that there exists, for every $\gamma\in\Gamma$, an isometry $u(\gamma)$ of H_0 onto $H(\gamma)$ such that $u(\gamma)\mathfrak{M}_0u(\gamma)^{-1}=\mathfrak{M}(\gamma)$. Then there exists an isometry of $L^2(\Gamma,\mu)\otimes H_0$ onto H which transforms the tensor product $\mathfrak{A}\otimes\mathfrak{M}_0$ into $\mathfrak{M}$.

PROOF. We may assume that $\gamma\to H(\gamma)$ is a constant field corresponding to H_0. Then $u(\gamma)$ is a unitary operator on H_0 such that $u(\gamma)\mathfrak{M}_0u(\gamma)^{-1}=\mathfrak{M}(\gamma)$ for every $\gamma\in\Gamma$. Let $\mathfrak{N}$ denote the set of all von Neumann algebras acting on H_0. By [4], $\mathfrak{N}$ has a standard Borel structure with the property: $\gamma \to \mathfrak{M}(\gamma) \in \mathfrak{A}$ is a measurable field if and only if $\gamma \to \mathfrak{M}(\gamma)$ is a $\mathfrak{A}$-valued measurable function over Γ. Define the action of the unitary group $\mathfrak{U}$ on $\mathfrak{A}$ by $u(\mathfrak{M}) = u\mathfrak{M}u^{-1}$ for $u \in \mathfrak{U}$ and $\mathfrak{M} \in \mathfrak{A}$. Then $\mathfrak{U}$ becomes a Borel transformation group of $\mathfrak{A}$ by [5]. Let $\mathfrak{U}_0$ denote the stability group of $\mathfrak{U}$ at $\mathfrak{M}_0$. Then $\mathfrak{U}_0$ is a closed subgroup of $\mathfrak{U}$. Therefore, the quotient map: $\mathfrak{U} \mapsto \mathfrak{U}/\mathfrak{U}_0$ has a Borel transversal $\mathfrak{W}$ as in Theorem 1. Putting $\Psi(u) = u(\mathfrak{M}_0)$ for $u \in \mathfrak{U}$, Ψ becomes a one-to-one Borel map of the standard Borel space $\mathfrak{W}$ into the standard Borel space $\mathfrak{A}$, so that Ψ is a Borel isomorphism of $\mathfrak{W}$ onto $\Psi(\mathfrak{W})$. Let Φ denote the inverse map of Ψ defined on $\Psi(\mathfrak{W})$. Putting $v(\gamma) = \Phi(\mathfrak{M}(\gamma))$ for every $\gamma \in \Gamma$, $\gamma \to v(\gamma)$ is a measurable $\mathfrak{U}$-valued function over Γ. Then

$$v = \int_\Gamma^\oplus v(\gamma)d\mu(\gamma)$$

is the desired isometry of $L^2(\Gamma, \mu) \otimes H_0$ onto H. This completes the proof.

Making use of Theorem 2, we can settle the problem proposed by S. Sakai in [7, Problem 4, p. 3.55] as follows:

THEOREM 3. *Let $\gamma \to \mathfrak{M}(\gamma)$ be a measurable field of von Neumann algebras over Γ. Suppose each von Neumann algebra $\mathfrak{M}(\gamma)$ is isomorphic to a fixed von Neumann algebra $\mathfrak{M}_0$. Then the direct integral*

$$\mathfrak{M} = \int_\Gamma^\oplus \mathfrak{M}(\gamma)d\mu(\gamma)$$

is isomorphic to the tensor product $\mathfrak{a} \otimes \mathfrak{M}_0$ of the diagonal algebra $\mathfrak{a}$ and $\mathfrak{M}_0$.

PROOF. Suppose $\gamma \mapsto H(\gamma)$ is a measurable field of Hilbert spaces over Γ associated with $\gamma \mapsto \mathfrak{M}(\gamma)$; that is, each $\mathfrak{M}(\gamma)$ acts on $H(\gamma)$. Let K_0 be a countably infinite dimensional Hilbert space. Then $\gamma \mapsto H(\gamma) \otimes K_0 = \tilde{H}(\gamma)$ is a measurable field of Hilbert spaces by [1, Proposition 11, p. 152]. Further, $\gamma \mapsto \mathfrak{M}(\gamma) \otimes I = \tilde{\mathfrak{M}}(\gamma)$ also becomes a measurable field of von Neumann algebras. Let H_0 denote the underlying Hilbert space of $\mathfrak{M}_0$. Putting $\tilde{\mathfrak{M}}_0 = \mathfrak{M}_0 \otimes I$ on $\tilde{H}_0 = H_0 \otimes K_0$, each $\tilde{\mathfrak{M}}(\gamma)$ is isomorphic to $\tilde{\mathfrak{M}}_0$. Since the commutant $\tilde{\mathfrak{M}}(\gamma)'$ of each $\tilde{\mathfrak{M}}(\gamma)$ becomes $\mathfrak{M}(\gamma)' \otimes \mathfrak{B}(K_0)$ which is properly infinite, each algebra $\tilde{\mathfrak{M}}(\gamma)$ is spatially isomorphic to $\tilde{\mathfrak{M}}_0$. Therefore, by Theorem 2, the direct integral

$$\tilde{\mathfrak{M}} = \int_\Gamma^\oplus \tilde{\mathfrak{M}}(\gamma)d\mu(\gamma)$$

is spatially isomorphic to $\mathfrak{A}\otimes\tilde{\mathfrak{M}}_0$. By [1, Proposition 3, p. 185], $\tilde{\mathfrak{M}}$ and $\mathfrak{A}\otimes\tilde{\mathfrak{M}}_0$ are spatially isomorphic to $\mathfrak{M}\otimes I$ and $(\mathfrak{A}\otimes\mathfrak{M}_0)\otimes I$ respectively, so that $\mathfrak{M}$ is isomorphic to $\mathfrak{A}\otimes\mathfrak{M}_0$. This completes the proof.

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