A GENERATOR FOR A SEMIGROUP OF NONLINEAR TRANSFORMATIONS

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Let $S$ be a finite-dimensional Hilbert space and $T$ a function from $[0, \infty)$ to the set of continuous transformations from $S$ to $S$ satisfying

Condition (1) $T(0) = I$, and $T(x)T(y) = T(x+y)$ if $x, y \geq 0$,

Condition (2) if $p$ is in $S$ and $g_p(x) = T(x)p$ for all $x \geq 0$, then $g_p$ is continuous,

Condition (3) for each $x \geq 0$, $T(x)$ is nonexpansive ($\|T(x)p - T(x)q\| \leq \|p - q\|$ for all $p$ and $q$ in $S$),

Condition (4) $S$ contains a rest point $r$ (i.e., $T(x)r = r$ for all $x \geq 0$).

For each $\delta > 0$, let $A_{\delta} = (1/\delta)[T(\delta) - I]$. For each $p$ in $S$ for which $\lim_{t \to 0} A_{\delta}p$ exists, let $A_{\delta}p = \lim_{t \to 0} A_{\delta}p$.

It is well known that even for $S$ infinite-dimensional, if the semigroup $\{T(x) | x \geq 0\}$ is linear (i.e., each $T(x)$ is linear), then the function $A$, called the infinitesimal generator of the semigroup, is defined on a dense subset of $S$, and for each $p$ in $S$, and each $x \geq 0$, $\lim_{n \to \infty} [I - (x/n)A]^{-n}p = T(x)p$. (See, for example, [1].)

In [2] and [3], J. W. Neuberger considered semigroups similar to the ones considered in this paper, with the following condition assumed:

Condition (5) there is a dense subset $D$ of $S$ such that if $p$ is in $D$, then the derivative $g_p'$ is continuous with domain $[0, \infty)$. In [3] he gave the following result, which will be used in a proof in this paper.

Theorem 1. Suppose $S$ is a normed linear complete space and Conditions (1), (2), (3), and (5) are satisfied. If $p$ is in $S$ and $x > 0$ and $\epsilon > 0$, there is a positive number $\delta$ such that if $0 < y \leq x$ and $0 = t_0 < t_1 < \cdots < t_{n+1} = y$ and $|t_{i+1} - t_i| < \delta$ for $i = 0, 1, 2, \cdots, n$, then

$$\limsup_{t_0, t_1, \ldots, t_n \to 0} \left\| \prod_{i=0}^{n} [I - (t_{i+1} - t_i)A_{t_i}]^{-1}p - T(y)p \right\| < \epsilon.$$  

The purpose of this paper is to define a set $\{I_x | x \geq 0\}$ of functions in terms of the functions $A_x$ in such a way that the functions $I_x$ generate the semigroup. The main results follow.

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1 This paper is part of the author's Doctoral dissertation at Emory University, prepared under the supervision of J. W. Neuberger.
Theorem 2. If \( \{ \delta_i \}_{i=1}^\infty \) is a sequence of positive numbers converging to 0, then there is a subsequence \( \{ \epsilon_i \}_{i=1}^\infty \) of \( \{ \delta_i \}_{i=1}^\infty \) such that if \( x \geq 0 \) and \( p \) is in \( S \), then \( \{(I-xA_\epsilon)^{-1}p\}_{i=1}^\infty \) converges to a point in \( S \). For such a sequence \( \{ \epsilon_i \}_{i=1}^\infty \), if \( x \geq 0 \), let \( I_x \) be the function from \( S \) into \( S \) defined by

\[
I_x p = \lim_{n \to \infty} (I - xA_{\epsilon_n})^{-1}p
\]

for each \( p \) in \( S \). Then each of the following is true.

(i) \( \|I_x p - I_x q\| \leq \|p - q\| \) for all \( x \geq 0 \) and all \( p \) and \( q \) in \( S \).

(ii) If \( x > 0 \), then \( \lim_{n \to x} I_x p = I_x p \).

(iii) If Condition (5) is satisfied, then \( \lim_{n \to \infty} (I_{y/n})^n p = T(y)p \) for all \( y > 0 \) and all \( p \) in \( S \).

This theorem may be compared to a result found recently by Shinnosuke Ôharu [4], who considered \( S \) to be a Banach space and assumed conditions which implied Conditions (1), (2), (3), and (5). For a nonlinear semigroup with these properties he found the following.

Theorem 3. Let \( \{T(x) | x \geq 0\} \) be a semigroup as described above, and let \( A \) be the infinitesimal generator such that for some \( x_0 > 0 \), the range of \( I - x_0 A \) is dense in \( S \). Then for every \( x > 0 \), there exist the function \( (I - xA)^{-1} \) and its unique extension \( L(x) \) onto \( S \), which is nonexpansive, and \( T(x) \) is represented by

\[
\lim_{n \to \infty} L(x/n)^n p = T(x)p
\]

for each \( x \geq 0 \) and each \( p \) in \( S \).

If a semigroup satisfies the assumptions for both Theorem 3 and (iii) of Theorem 2, then \( I_x = L(x) \) for \( x \geq 0 \). However, Theorem 2 does not assume that for some \( x > 0 \), the range of \( I - xA \) is dense in \( S \), and also Theorem 2 does not state that the collection \( \{I_x | x \geq 0\} \) is unique.

The proof of Theorem 2 will be developed by a sequence of lemmas, for which \( S \) is finite-dimensional and Conditions (1)-(4) are assumed.

Lemma 1. If \( p \) is in \( S \) and \( x > 0 \) and \( \{ \delta_i \}_{i=1}^\infty \) is a sequence of positive numbers, then there is a subsequence \( \{ \epsilon_i \}_{i=1}^\infty \) of \( \{ \delta_i \}_{i=1}^\infty \) such that \( \{(I-xA_\epsilon)^{-1}p\}_{i=1}^\infty \) converges to a point in \( S \).

Proof. In [2], Neuberger has a short proof that for \( x > 0 \) and \( \delta > 0 \), \( (I-xA_\delta)^{-1} \) exists, has domain \( S \), and is nonexpansive.

Now there is a rest point \( r \) in \( S \), and thus for \( \delta > 0 \), \( (I-xA_\delta)^{-1}r = r \). Thus
\[ \| r - (I - xA_s)^{-1}p \| = \| (I - xA_s)^{-1}r - (I - xA_s)^{-1}p \| \leq \| r - p \|. \]

Then the set \( \{(I - xA_s)^{-1}p \mid i = 1, 2, \ldots \} \) is bounded, and since \( S \) is finite-dimensional, the lemma follows.

**Lemma 2.** If \( x > 0 \) and \( \{\delta_i\}_{i=1}^\infty \) is a sequence of positive numbers converging to 0, then there is a subsequence \( \{e_i\}_{i=1}^\infty \) of \( \{\delta_i\}_{i=1}^\infty \) such that for all \( p \) in \( S \), \( \{(I - xA_{e_i})^{-1}p\}_{i=1}^\infty \) converges.

**Proof.** Let \( K = \{p_1, p_2, \ldots \} \) be a dense subset of \( S \). By Lemma 1, there is a subsequence \( \{\delta_{ii}\}_{i=1}^\infty \) of \( \{\delta_i\}_{i=1}^\infty \) such that \( \{(I - xA_{\delta_{ii}})^{-1}p_i\}_{i=1}^\infty \) converges. Continuing, for each \( n > 1 \), a subsequence \( \{\delta_{n1}\}_{i=1}^\infty \) of \( \{\delta_{(n-1)i}\}_{i=1}^\infty \) can be obtained such that \( \{(I - xA_{\delta_{n1}})^{-1}p_n\}_{i=1}^\infty \) converges. Consider the subsequence \( \{\delta_{ii}\}_{i=1}^\infty \) of \( \{\delta_{i\,i}\}_{i=1}^\infty \). It is easily seen that for each \( p_n \) in \( K \), \( \{(I - xA_{\delta_{n1}})^{-1}p_n\}_{i=1}^\infty \) converges to the sequential limit of \( \{(I - xA_{\delta_{ii}})^{-1}p_n\}_{i=1}^\infty \).

For each positive integer \( i \), let \( e_i = \delta_{ii} \). Then for all \( q \) in \( K \), \( \{(I - xA_{\delta_{ii}})^{-1}q\}_{i=1}^\infty \) converges. Suppose \( p \) is in \( S \). If \( e > 0 \), there is some \( q \) in \( K \) such that \( \| p - q \| < e/3 \). There is some positive integer \( N \) such that \( n > N \) and \( m > N \), then \( \| (I - xA_{e_n})^{-1}q - (I - xA_{e_m})^{-1}q \| < e/3 \). Since \( \| p - q \| < e/3 \), it follows that \( \| (I - xA_{e_n})^{-1}p - (I - xA_{e_m})^{-1}q \| < e/3 \) and \( \| (I - xA_{e_n})^{-1}p - (I - xA_{e_m})^{-1}q \| < e/3 \). It follows that \( \| (I - xA_{e_n})^{-1}p - (I - xA_{e_m})^{-1}p \| < e \), and thus \( \{(I - xA_{\delta_{ii}})^{-1}p\}_{i=1}^\infty \) is a Cauchy sequence and hence converges to a point in \( S \). The lemma is proved.

**Lemma 3.** Suppose \( Q \) is a countable subset of \( (0, \infty) \) and \( \{\delta_i\}_{i=1}^\infty \) is a sequence of positive numbers converging to 0. Then there is a subsequence \( \{e_i\}_{i=1}^\infty \) of \( \{\delta_i\}_{i=1}^\infty \) such that for each \( x \) in \( Q \) and each \( p \) in \( S \), \( \{(I - xA_{e_i})^{-1}p\}_{i=1}^\infty \) converges to a point in \( S \).

**Proof.** Let \( Q = \{x_1, x_2, \ldots \} \) be a subset of \( (0, \infty) \). By Lemma 2, there is a subsequence \( \{\delta_{i1}\}_{i=1}^\infty \) of \( \{\delta_i\}_{i=1}^\infty \) such that \( \{(I - x_{i1}A_i)^{-1}p\}_{i=1}^\infty \) converges for all \( p \) in \( S \). Then there is a subsequence \( \{\delta_{i2}\}_{i=1}^\infty \) of \( \{\delta_{i1}\}_{i=1}^\infty \) such that \( \{(I - x_{i2}A_{e_{i1}})^{-1}p\}_{i=1}^\infty \) converges for all \( p \) in \( S \). By continuing this process, which is similar to that used in the proof of Lemma 2, one can show that the subsequence \( \{\delta_{ii}\}_{i=1}^\infty \) of \( \{\delta_{i\,i}\}_{i=1}^\infty \) has the property that for every \( x \) in \( Q \) and every \( p \) in \( S \), \( \{(I - xA_{\delta_{ii}})^{-1}p\}_{i=1}^\infty \) converges.

**Lemma 4.** Suppose \( p \) is in \( S \). If \( \delta > 0 \), let \( F_\delta \) be the function from \( [0, \infty) \) into \( S \) defined by \( F_\delta(x) = (I - xA_s)^{-1}p \). Then \( F_\delta \) is continuous.

**Proof.** Let \( x \geq 0 \). Let \( M \) be a positive number greater than \( \|A_s(I - xA_s)^{-1}p\| \). Let \( \epsilon > 0 \). Let \( y \geq 0 \) such that \( y - x < \epsilon/M \). Then
\[ \|F_a(y) - F_a(x)\| = \|(I - yA_a)^{-1}p - (I - xA_a)^{-1}p\| \]
\[ = \|(I - yA_a)^{-1}p - (I - yA_a)^{-1}(I - yA_a)(I - xA_a)^{-1}p\| \]
\[ \leq \|p - (I - yA_a)(I - xA_a)^{-1}p\| \]
\[ = \|(I - xA_a)(I - xA_a)^{-1}p - (I - yA_a)(I - xA_a)^{-1}p\| \]
\[ = \|y - x\| \|A_a(I - xA_a)^{-1}p\| < \epsilon. \]

**Lemma 5.** If \( p \) is in \( S \) and \( c > 0 \), then the set
\[ \{\|A_a(I - xA_a)^{-1}p\| : x \geq c, \delta > 0\} \]

is bounded above.

**Proof.** From the proof of Lemma 1, it is easily seen that the set
\[ \{\|y - x\| A_a(I - xA_a)^{-1}p\| : x \geq 0, \delta > 0\} \]
is bounded above. Let \( M > 0 \) be an upper bound to this set. If \( x \geq c \) and \( \delta > 0 \), then
\[ \|A_a(I - xA_a)^{-1}p\| \]
\[ = \left(\frac{1}{x}\right)\|y - x\| A_a(I - xA_a)^{-1}p\| \]
\[ = \left(\frac{1}{x}\right)\|(I - xA_a)(I - xA_a)^{-1}p - (I - xA_a)^{-1}p\| \]
\[ = \left(\frac{1}{x}\right)\|p - (I - xA_a)^{-1}p\| \leq \left(\frac{1}{c}\right)\|p\| + \|p\| \leq \left(\frac{1}{c}\right)\|p\| + M. \]

**Lemma 6.** Suppose \( p \) is in \( S \) and for each \( x \geq 0 \), \( F_a(x) = (I - xA_a)^{-1}p \). Then the set \( \{F_a : \delta > 0\} \) is equicontinuous on \((0, \infty)\).

**Proof.** Let \( x > 0 \). By Lemma 5, there is an upper bound \( M > 0 \) to the set \( \{|y - x| A_a(I - xA_a)^{-1}p\| : x \geq 0, \delta > 0\} \). Let \( \epsilon > 0 \). If \( y \) is a positive number such that \(|x - y| < \epsilon/M\), then by the same argument as that used in the proof of Lemma 4, \( \|F_a(x) - F_a(y)\| < \epsilon \) for all \( \delta > 0 \).

**Lemma 7.** If \( \{\delta_i\}_{i=1}^\infty \) is a sequence of positive numbers converging to 0, there is a subsequence \( \{\epsilon_i\}_{i=1}^\infty \) of \( \{\delta_i\}_{i=1}^\infty \) such that for each \( x \geq 0 \) and each \( p \) in \( S \), \( \{(I - xA_a)^{-1}p\}_{i=1}^\infty \) converges to a point in \( S \).

**Proof.** Let \( Q \) be a countable dense subset of \((0, \infty)\). By Lemma 3, there is a subsequence \( \{\epsilon_i\}_{i=1}^\infty \) of \( \{\delta_i\}_{i=1}^\infty \) such that for all \( x \) in \( Q \) and all \( p \) in \( S \), \( \{(I - xA_a)^{-1}p\}_{i=1}^\infty \) converges. It will be shown that for all \( x \geq 0 \) and all \( p \) in \( S \), this sequence converges.

For each \( q \) in \( S \) and each positive integer \( i \), let
\[ F_{\epsilon_i}(x) = (I - xA_{\epsilon_i})^{-1}q \]
for all \( x \geq 0 \). Let \( p \) be in \( S \). From Lemma 6, the set \( \{F_{\epsilon_i} : i = 1, 2, \ldots\} \) is equicontinuous on \((0, \infty)\). Also, for each \( z \) in the dense subset \( Q \)
of \((0, \infty)\), the sequence \[ \{ F_{p,i}(x) \}_{i=1}^\infty \] converges. Thus it follows that for every \(x\) in \((0, \infty)\), the sequence \[ \{ F_{p,i}(x) \}_{i=1}^\infty \] is a Cauchy sequence and hence converges to some point in \(S\). That is, for every \(x > 0\), \[ \{ (I - xA_{e_i})^{-1}p \}_{i=1}^\infty \] converges. Since for \(x = 0\) the sequence clearly converges, the lemma is true, and the first statement in Theorem 2 is proved.

Consider the sequence \( \{ \epsilon_i \}_{i=1}^\infty \) given in the proof of Lemma 7 and consider the functions \( F_{p,i} \) defined in the proof. For each \(x \geq 0\) and each \(p\) in \(S\), let \( \tau_xp \) be the sequential limit of \( \{ (7 - xA_{e_i})^{-1} \} P \}_{i=1}^\infty \). For each \(p\) in \(S\), let \( G_p \) be the function from \([0, \infty)\) into \(S\) defined by \( G_p(x) = \tau_xp \), so that for each \(x \geq 0\), \( G_p(x) \) is the sequential limit of \( \{ F_{p,i}(x) \}_{i=1}^\infty \). These functions will be used in Lemmas 8 and 9 below.

**Lemma 8.** For each \(x \geq 0\), \( I_x \) is nonexpansive.

**Proof.** Let \(x \geq 0\), and let each of \(p\) and \(q\) be in \(S\). If \(\epsilon > 0\) there is a positive integer \(n\) such that \[ \| (I - xA_{e_n})^{-1}p - \tau_xp \| < \epsilon/2 \] and \[ \| (I - xA_{e_n})^{-1}q - \tau_xq \| < \epsilon/2. \] From this it follows that \[ \| I_xp - \tau_xq \| < \| (I - xA_{e_n})^{-1}p - (I - xA_{e_n})^{-1}q \| + \epsilon. \] Since \( (I - xA_{e_n})^{-1} \) is nonexpansive, it follows that \( \| I_xp - \tau_xq \| < \| p - q \| + \epsilon. \) Thus \( I_x \) is nonexpansive, and (i) of Theorem 2 is proved.

**Lemma 9.** For each \(p\) in \(S\), \( G_p \) is continuous on \((0, \infty)\).

**Proof.** Suppose \(p\) is in \(S\). Since for each \(x > 0\), \( G_p(x) \) is the sequential limit of \( \{ F_{p,i}(x) \}_{i=1}^\infty \), and since the set \{ \( F_{p,i} \) \( i = 1, 2, \ldots \) \} is equicontinuous on \((0, \infty)\), it follows that \( G_p \) is continuous on \((0, \infty)\).

Since for each \(p\) in \(S\) and each \(x \geq 0\), \( I_xp = G_p(x) \), it follows from Lemma 9 that for \(x > 0\) and \(p\) in \(S\), \( \lim_{y \to x} I_yp = \tau_xp \), and so (ii) of Theorem 2 is proved.

All of the above lemmas would still be true if, instead of assuming that \(S\) contains a rest point, it is assumed that for each point \(p\) in \(S\), \[ \{ \| (I - xA_{e_i})^{-1}p \| : \delta > 0, \ x \geq 0 \} \] is bounded above. Requiring this boundedness is a weaker condition than requiring a rest point, since if \(S\) contains a rest point, the above set is bounded above. However, to assume \(S\) contains a rest point does not appear to be a very strong condition, since if there is at least one point \(p\) in \(S\) such that \( g_p \) is not one-to-one, then \(S\) contains a rest point. For if \( g_p(u) = g_p(v) \) where \(0 \leq u < v\), then setting \( q = g_p(u) \) it follows that \[ g_q(v - u) = T(v - u)q = T(v - u)g_p(u) \] \[ = T(v - u)T(u)p = T(v)p = g_p(v) = q. \]
Then, since \( g_\varphi(v-u) = q \) and \( v-u > 0 \), \( g_\varphi \) is said to be periodic of period \( v-u \), and the following result \([5]\) of the author can be used to show that \( (1/(v-u))\int_0^{v-u} g_\varphi \) is a rest point.

**Theorem 4.** Assume \( S \) is a Hilbert space and Conditions (1), (2), and (3) are satisfied. If for some point \( p \) in \( S \), \( g_\varphi \) is periodic of period \( \varepsilon > 0 \), then \( (1/\varepsilon)\int_0^\varepsilon g_\varphi \) is a rest point.

For Lemma 10, Condition (5) will also be assumed. For Lemma 10, let \( E \) denote a sequence \( \{\varepsilon_i\}_{i=1}^\infty \) of positive numbers converging to 0 such that for every \( x \geq 0 \) and every \( p \) in \( S \), \( \{(I-\alpha A_{\epsilon_i})^{-1}p\}_{i=1}^\infty \) converges, and let \( I_{\alpha}p \) be the sequential limit. Then, by Lemma 8, for each \( x \geq 0 \), \( I_x \) is nonexpansive.

**Lemma 10.** Suppose \( y > 0 \) and \( p \) is in \( S \). If \( \varepsilon > 0 \), there is a \( \delta > 0 \) such that if \( 0=t_0 < t_1 < \cdots < t_{n+1} = y \) and \( |t_{i+1} - t_i| < \delta \) for \( i = 0, 1, \ldots, n \), then

\[
\left\| \prod_{i=0}^{n} I_{t_{i+1} - t_i} p - T(y)p \right\| < \varepsilon.
\]

**Proof.** Let \( \varepsilon > 0 \). Using Theorem 1, let \( \delta \) be a positive number such that if \( 0=t_0 < t_1 < \cdots < t_{n+1} = y \) and \( |t_{i+1} - t_i| < \delta \) for \( i = 0, 1, 2, \ldots, n \), then

\[
\limsup_{\delta_0, \delta_1, \ldots, \delta_n \to 0} \left\| \prod_{i=0}^{n} (I - (t_{i+1} - t_i) A_{\delta_i})^{-1} p - T(y) p \right\| < \varepsilon/2.
\]

Take \( 0=t_0 < t_1 < \cdots < t_{n+1} = y \) where \( |t_{i+1} - t_i| < \delta \) for \( i = 0, 1, \ldots, n \). There is a positive number \( k \) such that if each of \( \delta_0, \delta_1, \ldots, \delta_n \) is less than \( k \), then

\[
\left\| \prod_{i=0}^{n} (I - (t_{i+1} - t_i) A_{\delta_i})^{-1} p - T(y)p \right\| < \varepsilon/2.
\]

Now \( I_{t_{n+1} - t_n} p = \lim_{m \to \infty} (I - (t_{n+1} - t_n) A_{\delta_m})^{-1} p \), and so \( \delta_n \) can be chosen to be a number in the sequence \( E \) which is less than \( k \) and such that

\[
\left\| (I - (t_{n+1} - t_n) A_{\delta_n})^{-1} p - I_{t_{n+1} - t_n} p \right\| < \varepsilon/(n + 1).
\]

Since \( I_{t_n - t_{n-1}} \) is nonexpansive, it follows that

(1) \[
\left\| I_{t_n - t_{n-1}} (I - (t_{n+1} - t_n) A_{\delta_n})^{-1} p - I_{t_n - t_{n-1}} I_{t_{n+1} - t_n} p \right\| < \varepsilon/2(n + 1).
\]

Now

\[
I_{t_n - t_{n-1}} (I - (t_{n+1} - t_n) A_{\delta_n})^{-1} p = \lim_{m \to \infty} (I - (t_n - t_{n-1}) A_{\delta_n})^{-1} (I - (t_{n+1} - t_n) A_{\delta_n})^{-1} p
\]
and so $\delta_{n-1}$ can be chosen to be a number in sequence $E$ which is less than $k$ and such that
\begin{equation}
\| [I - (t_n - t_{n-1}) A_{\delta_{n-1}}]^{-1} [I - (t_{n+1} - t_n) A_{\delta_n}]^{-1} p - I_{t_{n-1}} [I - (t_{n+1} - t_n) A_{\delta_n}]^{-1} p \| < \epsilon/2(n + 1).
\end{equation}

From (1) and (2) it follows that
\begin{align*}
\| I_{t_{n-1}} I_{t_{n+1} - t_n} p - [I - (t_n - t_{n-1}) A_{\delta_{n-1}}]^{-1} [I - (t_{n+1} - t_n) A_{\delta_n}]^{-1} p \| & < 2\epsilon/(n + 1).
\end{align*}

Since $I_{t_{n-1} - t_{n-2}}$ is nonexpansive, it follows that
\begin{align*}
\| I_{t_{n-1} - t_{n-2}} I_{t_{n+1} - t_n} I_{t_{n+1} - t_n} p - I_{t_{n-1} - t_{n-2}} [I - (t_n - t_{n-1}) A_{\delta_{n-1}}]^{-1} [I - (t_{n+1} - t_n) A_{\delta_n}]^{-1} p \| & < 2\epsilon/(n + 1).
\end{align*}

Now
\begin{align*}
I_{t_{n-1} - t_{n-2}} [I - (t_n - t_{n-1}) A_{\delta_{n-1}}]^{-1} [I - (t_{n+1} - t_n) A_{\delta_n}]^{-1} p &= \lim_{m \to \infty} [I - (t_{n-1} - t_{n-2}) A_{\epsilon_{n_m}}]^{-1} [I - (t_{n-1} - t_{n-2}) A_{\delta_{n-1}}]^{-1} \\
&\quad \cdot [I - (t_{n+1} - t_n) A_{\delta_n}]^{-1} p.
\end{align*}

Let $\delta_{n-2}$ be a number in sequence $E$ which is less than $k$ and such that
\begin{align*}
\| [I - (t_{n-1} - t_{n-2}) A_{\delta_{n-2}}]^{-1} [I - (t_n - t_{n-1}) A_{\delta_{n-1}}]^{-1} [I - (t_{n+1} - t_n) A_{\delta_n}]^{-1} p - I_{t_{n-1} - t_{n-1}} [I - (t_n - t_{n-1}) A_{\delta_{n-1}}]^{-1} [I - (t_{n+1} - t_n) A_{\delta_n}]^{-1} p \| & < \epsilon/2(n + 1).
\end{align*}

From (3) and (4) it follows that
\begin{align*}
\| I_{t_{n-1} - t_{n-2}} I_{t_{n-1} - t_{n-1}} I_{t_{n+1} - t_n} p - [I - (t_{n-1} - t_{n-2}) A_{\delta_{n-2}}]^{-1} \\
&\quad \cdot [I - (t_n - t_{n-1}) A_{\delta_{n-1}}]^{-1} [I - (t_{n+1} - t_n) A_{\delta_n}]^{-1} p \| < 3\epsilon/2(n + 1).
\end{align*}

Continuing this process of choosing $\delta_n, \delta_{n-1}, \ldots, \delta_0$, it can be seen that
\begin{align*}
\left\| \prod_{i=0}^{n} I_{t_{i+1} - t_i} p - \prod_{i=0}^{n} [I - (t_{i+1} - t_i) A_{\delta_i}]^{-1} p \right\| & < (n + 1)\epsilon/2(n + 1).
\end{align*}

But each of $\delta_0, \delta_1, \ldots, \delta_n$ is less than $k$ and thus
\begin{align*}
\left\| \prod_{i=0}^{n} [I - (t_{i+1} - t_i) A_{\delta_i}]^{-1} p - T(\gamma) p \right\| & < \epsilon/2.
\end{align*}
From (5) and (6) it follows that
\[ \left\| \prod_{i=0}^{n} I_{t_{i+1} - t_{i}} p - T(y)p \right\| < \epsilon, \]
and the lemma is proved.

It follows from Lemma 10 that \( \lim_{n \to \infty} (I_{y/n})^n p = T(y)p \), and thus (iii) of Theorem 2 is proved.

References


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