A GENERATOR FOR A SEMIGROUP OF NONLINEAR TRANSFORMATIONS

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Let S be a finite-dimensional Hilbert space and T a function from $[0, \infty)$ to the set of continuous transformations from S to S satisfying

Condition (1) $T(0)=I$, and $T(x)T(y)=T(x+y)$ if $x, y \geqq 0$,

Condition (2) if p is in S and $g_p(x)=T(x)p$ for all $x \geqq 0$, then g_p is continuous,

Condition (3) for each $x \geqq 0$, $T(x)$ is nonexpansive ($\|T(x)p-T(x)q\| \leqq \|p-q\|$ for all p and q in S),

Condition (4) S contains a rest point r (i.e., $T(x)r=r$ for all $x \geqq 0$).

For each $\delta>0$, let $A_\delta=(1/\delta)[T(\delta)-I]$. For each p in S for which $\lim_{\delta\to 0} A_\delta p$ exists, let $Ap=\lim_{\delta\to 0} A_\delta p$.

It is well known that even for S infinite-dimensional, if the semigroup $\{T(x)|x \geqq 0\}$ is linear (i.e., each $T(x)$ is linear), then the function A, called the infinitesimal generator of the semigroup, is defined on a dense subset of S, and for each p in S, and each $x \geqq 0$, $\lim_{n\to\infty} [I-(x/n)A]^{-n}p=T(x)p$. (See, for example, [1].)

In [2] and [3], J. W. Neuberger considered semigroups similar to the ones considered in this paper, with the following condition assumed:

Condition (5) there is a dense subset D of S such that if p is in D, then the derivative g_p' is continuous with domain $[0, \infty)$. In [3] he gave the following result, which will be used in a proof in this paper.

THEOREM 1. *Suppose S is a normed linear complete space and Conditions* (1), (2), (3), *and* (5) *are satisfied. If p is in S and $x>0$ and $\epsilon>0$, there is a positive number δ such that if $0<y\leqq x$ and $0=t_0<t_1<\cdots<t_{n+1}=y$ and $|t_{i+1}-t_i|<\delta$ for $i=0, 1, 2, \cdots, n$, then*

$$\limsup_{\delta_0,\delta_1,\cdots,\delta_n\to 0}\left\|\prod_{i=0}^{n}[I-(t_{i+1}-t_i)A_{\delta_i}]^{-1}p-T(y)p\right\|<\epsilon.$$

The purpose of this paper is to define a set $\{I_x|x\geqq 0\}$ of functions in terms of the functions A_δ in such a way that the functions I_x generate the semigroup. The main results follow.

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THEOREM 2. *If* $\{\delta_i\}_{i=1}^{\infty}$ *is a sequence of positive numbers converging to* 0, *then there is a subsequence* $\{\epsilon_i\}_{i=1}^{\infty}$ *of* $\{\delta_i\}_{i=1}^{\infty}$ *such that if* $x \geqq 0$ *and* p *is in* S, *then* $\{(I-xA_{\epsilon_i})^{-1}p\}_{i=1}^{\infty}$ *converges to a point in* S. *For such a sequence* $\{\epsilon_i\}_{i=1}^{\infty}$, *if* $x \geqq 0$, *let* I_x *be the function from* S *into* S *defined by*

$$I_x p = \lim_{n\to\infty} (I - xA_{\epsilon_n})^{-1} p$$

for each p *in* S. *Then each of the following is true.*

(i) $\|I_x p - I_x q\| \leqq \|p-q\|$ *for all* $x \geqq 0$ *and all* p *and* q *in* S.

(ii) *If* $x>0$, *then* $\lim_{y\to x} I_y p = I_x p$.

(iii) *If Condition* (5) *is satisfied, then* $\lim_{n\to\infty} (I_{y/n})^n p = T(y)p$ *for all* $y>0$ *and all* p *in* S.

This theorem may be compared to a result found recently by Shinnosuke Ôharu [4], who considered S to be a Banach space and assumed conditions which implied Conditions (1), (2), (3), and (5). For a nonlinear semigroup with these properties he found the following.

THEOREM 3. *Let* $\{T(x) \mid x \geqq 0\}$ *be a semigroup as described above, and let* A *be the infinitesimal generator such that for some* $x_0>0$, *the range of* $I-x_0A$ *is dense in* S. *Then for every* $x>0$, *there exist the function* $(I-xA)^{-1}$ *and its unique extension* $L(x)$ *onto* S, *which is nonexpansive, and* $T(x)$ *is represented by*

$$\lim_{n\to\infty} L(x/n)^n p = T(x)p$$

for each $x \geqq 0$ *and each* p *in* S.

If a semigroup satisfies the assumptions for both Theorem 3 and (iii) of Theorem 2, then $I_x = L(x)$ for $x \geqq 0$. However, Theorem 2 does not assume that for some $x>0$, the range of $I-xA$ is dense in S, and also Theorem 2 does not state that the collection $\{I_x \mid x \geqq 0\}$ is unique.

The proof of Theorem 2 will be developed by a sequence of lemmas, for which S is finite-dimensional and Conditions (1)–(4) are assumed.

LEMMA 1. *If* p *is in* S *and* $x>0$ *and* $\{\delta_i\}_{i=1}^{\infty}$ *is a sequence of positive numbers, then there is a subsequence* $\{\epsilon_i\}_{i=1}^{\infty}$ *of* $\{\delta\}_{i=1}^{\infty}$ *such that* $\{(I-xA_{\epsilon_i})^{-1}p\}_{i=1}^{\infty}$ *converges to a point in* S.

PROOF. In [2], Neuberger has a short proof that for $x>0$ and $\delta>0$, $(I-xA_\delta)^{-1}$ exists, has domain S, and is nonexpansive.

Now there is a rest point r in S, and thus for $\delta>0$, $(I-xA_\delta)^{-1}r=r$. Thus

$$\|r-(I-xA_\delta)^{-1}p\| = \|(I-xA_\delta)^{-1}r-(I-xA_\delta)^{-1}p\| \leqq \|r-p\|.$$

Then the set $\{(I-xA_{\delta_i})^{-1}p \mid i=1, 2, \cdots\}$ is bounded, and since S is finite-dimensional, the lemma follows.

LEMMA 2. *If $x>0$ and $\{\delta_i\}_{i=1}^\infty$ is a sequence of positive numbers converging to 0, then there is a subsequence $\{\epsilon_i\}_{i=1}^\infty$ of $\{\delta_i\}_{i=1}^\infty$ such that for all p in S, $\{(I-xA_{\epsilon_i})^{-1}p\}_{i=1}^\infty$ converges.*

PROOF. Let $K=\{p_1, p_2, \cdots\}$ be a dense subset of S. By Lemma 1, there is a subsequence $\{\delta_{1i}\}_{i=1}^\infty$ of $\{\delta_i\}_{i=1}^\infty$ such that $\{(I-xA_{\delta_{1i}})^{-1}p_1\}_{i=1}^\infty$ converges. Continuing, for each $n>1$, a subsequence $\{\delta_{ni}\}_{i=1}^\infty$ of $\{\delta_{(n-1)i}\}_{i=1}^\infty$ can be obtained such that $\{(I-xA_{\delta_{ni}})^{-1}p_n\}_{i=1}^\infty$ converges. Consider the subsequence $\{\delta_{ii}\}_{i=1}^\infty$ of $\{\delta_i\}_{i=1}^\infty$. It is easily seen that for each p_n in K, $\{(I-xA_{\delta_{ii}})^{-1}p_n\}_{i=1}^\infty$ converges to the sequential limit of $\{(I-xA_{\delta_{ni}})^{-1}p_n\}_{i=1}^\infty$.

For each positive integer i, let $\epsilon_i = \delta_{ii}$. Then for all q in K, $\{(I-xA_{\epsilon_i})^{-1}q\}_{i=1}^\infty$ converges. Suppose p is in S. If $\epsilon>0$, there is some q in K such that $\|p-q\|<\epsilon/3$. There is some positive integer N such that if $n>N$ and $m>N$, then $\|(I-xA_{\epsilon_n})^{-1}q-(I-xA_{\epsilon_m})^{-1}q\|<\epsilon/3$. Since $\|p-q\|<\epsilon/3$, it follows that $\|(I-xA_{\epsilon_n})^{-1}p-(I-xA_{\epsilon_n})^{-1}q\|<\epsilon/3$ and $\|(I-xA_{\epsilon_m})^{-1}p-(I-xA_{\epsilon_m})^{-1}q\|<\epsilon/3$. It follows that $\|(I-xA_{\epsilon_n})^{-1}p-(I-xA_{\epsilon_m})^{-1}p\|<\epsilon$, and thus $\{(I-xA_{\epsilon_i})^{-1}p\}_{i=1}^\infty$ is a Cauchy sequence and hence converges to a point in S. The lemma is proved.

LEMMA 3. *Suppose Q is a countable subset of $(0, \infty)$ and $\{\delta_i\}_{i=1}^\infty$ is a sequence of positive numbers converging to 0. Then there is a subsequence $\{\epsilon_i\}_{i=1}^\infty$ of $\{\delta_i\}_{i=1}^\infty$ such that for each x in Q and each p in S, $\{(I-xA_{\epsilon_i})^{-1}p\}_{i=1}^\infty$ converges to a point in S.*

PROOF. Let $Q=\{x_1, x_2, \cdots\}$ be a subset of $(0, \infty)$. By Lemma 2, there is a subsequence $\{\delta_{1i}\}_{i=1}^\infty$ of $\{\delta_i\}_{i=1}^\infty$ such that $\{(I-x_1A_{\delta_{1i}})^{-1}p\}_{i=1}^\infty$ converges for all p in S. Then there is a subsequence $\{\delta_{2i}\}_{i=1}^\infty$ of $\{\delta_{1i}\}_{i=1}^\infty$ such that $\{(I-x_2A_{\delta_{2i}})^{-1}p\}_{i=1}^\infty$ converges for all p in S. By continuing this process, which is similar to that used in the proof of Lemma 2, one can show that the subsequence $\{\delta_{ii}\}_{i=1}^\infty$ of $\{\delta_i\}_{i=1}^\infty$ has the property that for every x in Q and every p in S, $\{(I-xA_{\delta_{ii}})^{-1}p\}_{i=1}^\infty$ converges.

LEMMA 4. *Suppose p is in S. If $\delta>0$, let F_δ be the function from $[0, \infty)$ into S defined by $F_\delta(x)=(I-xA_\delta)^{-1}p$. Then F_δ is continuous.*

PROOF. Let $x\geqq 0$. Let M be a positive number greater than $\|A_\delta(I-xA_\delta)^{-1}p\|$. Let $\epsilon>0$. Let $y\geqq 0$ such that $|y-x|<\epsilon/M$. Then

$$\begin{aligned} \|F_\delta(y) - F_\delta(x)\| &= \|(I - yA_\delta)^{-1}p - (I - xA_\delta)^{-1}p\| \\ &= \|(I - yA_\delta)^{-1}p - (I - yA_\delta)^{-1}(I - yA_\delta)(I - xA_\delta)^{-1}p\| \\ &\leqq \|p - (I - yA_\delta)(I - xA_\delta)^{-1}p\| \\ &= \|(I - xA_\delta)(I - xA_\delta)^{-1}p - (I - yA_\delta)(I - xA_\delta)^{-1}p\| \\ &= |y - x| \, \|A_\delta(I - xA_\delta)^{-1}p\| < \epsilon. \end{aligned}$$

LEMMA 5. *If p is in S and $c>0$, then the set*

$$\{\|A_\delta(I - xA_\delta)^{-1}p\| \mid x \geqq c, \delta > 0\}$$

is bounded above.

PROOF. From the proof of Lemma 1, it is easily seen that the set $\{\|(I-xA_\delta)^{-1}p\| \mid x\geqq 0, \delta>0\}$ is bounded above. Let $M>0$ be an upper bound to this set. If $x\geqq c$ and $\delta>0$, then

$$\begin{aligned} &\|A_\delta(I - xA_\delta)^{-1}p\| \\ &\quad = (1/x)\| - xA_\delta(I - xA_\delta)^{-1}p\| \\ &\quad = (1/x)\|(I - xA_\delta)(I - xA_\delta)^{-1}p - (I - xA_\delta)^{-1}p\| \\ &\quad = (1/x)\|p - (I - xA_\delta)^{-1}p\| \leqq (1/c)[\|p\| + \|(I - xA_\delta)^{-1}p\|] \\ &\quad \leqq (1/c)[\|p\| + M]. \end{aligned}$$

LEMMA 6. *Suppose p is in S and for each $x\geqq 0$, $F_\delta(x)=(I-xA_\delta)^{-1}p$. Then the set $\{F_\delta \mid \delta>0\}$ is equicontinuous on $(0, \infty)$.*

PROOF. Let $x>0$. By Lemma 5, there is an upper bound $M>0$ to the set $\{\|A_\delta(I-xA_\delta)^{-1}p\| \mid \delta>0\}$. Let $\epsilon>0$. If y is a positive number such that $|x-y| <\epsilon/M$, then by the same argument as that used in the proof of Lemma 4, $\|F_\delta(x)-F_\delta(y)\| <\epsilon$ for all $\delta>0$.

LEMMA 7. *If $\{\delta_i\}_{i=1}^\infty$ is a sequence of positive numbers converging to 0, there is a subsequence $\{\epsilon_i\}_{i=1}^\infty$ of $\{\delta_i\}_{i=1}^\infty$ such that for each $x\geqq 0$ and each p in S, $\{(I-xA_{\epsilon_i})^{-1}p\}_{i=1}^\infty$ converges to a point in S.*

PROOF. Let Q be a countable dense subset of $(0,\infty)$. By Lemma 3, there is a subsequence $\{\epsilon_i\}_{i=1}^\infty$ of $\{\delta_i\}_{i=1}^\infty$ such that for all x in Q and all p in S, $\{(I-xA_{\epsilon_i})^{-1}p\}_{i=1}^\infty$ converges. It will be shown that for all $x\geqq 0$ and all p in S, this sequence converges.

For each q in S and each positive integer i, let

$$F_{q,i}(x) = (I - xA_{\epsilon_i})^{-1}q$$

for all $x\geqq 0$. Let p be in S. From Lemma 6, the set $\{F_{p,i} \mid i=1, 2, \cdots\}$ is equicontinuous on $(0, \infty)$. Also, for each z in the dense subset Q

of $(0, \infty)$, the sequence $\{F_{p,i}(z)\}_{i=1}^{\infty}$ converges. Thus it follows that for every x in $(0, \infty)$, the sequence $\{F_{p,i}(x)\}_{i=1}^{\infty}$ is a Cauchy sequence and hence converges to some point in S. That is, for every $x>0$, $\{(I-xA_{\epsilon_i})^{-1}p\}_{i=1}^{\infty}$ converges. Since for $x=0$ the sequence clearly converges, the lemma is true, and the first statement in Theorem 2 is proved.

Consider the sequence $\{\epsilon_i\}_{i=1}^{\infty}$ given in the proof of Lemma 7 and consider the functions $F_{q,i}$ defined in the proof. For each $x\geqq 0$ and each p in S, let $I_x p$ be the sequential limit of $\{(I-xA_{\epsilon_i})^{-1}p\}_{i=1}^{\infty}$. For each p in S, let G_p be the function from $[0, \infty)$ into S defined by $G_p(x)=I_x p$, so that for each $x\geqq 0$, $G_p(x)$ is the sequential limit of $\{F_{p,i}(x)\}_{i=1}^{\infty}$. These functions will be used in Lemmas 8 and 9 below.

LEMMA 8. *For each $x\geqq 0$, I_x is nonexpansive.*

PROOF. Let $x\geqq 0$, and let each of p and q be in S. If $\epsilon>0$ there is a positive integer n such that $\|(I-xA_{\epsilon_n})^{-1}p - I_x p\| < \epsilon/2$ and $\|(I-xA_{\epsilon_n})^{-1}q-I_x q\|<\epsilon/2$. From this it follows that

$$\|I_x p - I_x q\| < \|(I - xA_{\epsilon_n})^{-1}p - (I - xA_{\epsilon_n})^{-1}q\| + \epsilon.$$

Since $(I-xA_{\epsilon_n})^{-1}$ is nonexpansive, it follows that $\|I_x p-I_x q\| <\|p-q\| +\epsilon$. Thus I_x is nonexpansive, and (i) of Theorem 2 is proved.

LEMMA 9. *For each p in S, G_p is continuous on* $(0, \infty)$.

PROOF. Suppose p is in S. Since for each $x>0$, $G_p(x)$ is the sequential limit of $\{F_{p,i}(x)\}_{i=1}^{\infty}$, and since the set $\{F_{p,i} | i=1, 2, \cdots\}$ is equicontinuous on $(0, \infty)$, it follows that G_p is continuous on $(0, \infty)$.

Since for each p in S and each $x\geqq 0$, $I_x p=G_p(x)$, it follows from Lemma 9 that for $x>0$ and p in S, $\lim_{y\to x} I_y p=I_x p$, and so (ii) of Theorem 2 is proved.

All of the above lemmas would still be true if, instead of assuming that S contains a rest point, it is assumed that for each point p in S, $\{\|(I-xA_\delta)^{-1}p\| \mid \delta>0,\ x\geqq 0\}$ is bounded above. Requiring this boundedness is a weaker condition than requiring a rest point, since if S contains a rest point, the above set is bounded above. However, to assume S contains a rest point does not appear to be a very strong condition, since if there is at least one point p in S such that g_p is not one-to-one, then S contains a rest point. For if $g_p(u)=g_p(v)$ where $0\leqq u<v$, then setting $q=g_p(u)$ it follows that

$$\begin{aligned} g_q(v - u) &= T(v - u)q = T(v - u)g_p(u) \\ &= T(v - u)T(u)p = T(v)p = g_p(v) = q. \end{aligned}$$

Then, since $g_q(v-u)=q$ and $v-u>0$, g_q is said to be periodic of period $v-u$, and the following result [5] of the author can be used to show that $(1/(v-u))\int_0^{v-u}g_q$ is a rest point.

THEOREM 4. *Assume S is a Hilbert space and Conditions* (1), (2), *and* (3) *are satisfied. If for some point p in S, g_p is periodic of period $\epsilon>0$, then $(1/\epsilon)\int_0^\epsilon g_p$ is a rest point.*

For Lemma 10, Condition (5) will also be assumed. For Lemma 10, let E denote a sequence $\{\epsilon_i\}_{i=1}^\infty$ of positive numbers converging to 0 such that for every $x\geqq 0$ and every p in S, $\{(I-xA_{\epsilon_i})^{-1}p\}_{i=1}^\infty$ converges, and let I_xp be the sequential limit. Then, by Lemma 8, for each $x\geqq 0$, I_x is nonexpansive.

LEMMA 10. *Suppose $y>0$ and p is in S. If $\epsilon>0$, there is a $\delta>0$ such that if $0=t_0<t_1<t_2<\cdots<t_{n+1}=y$ and $|t_{i+1}-t_i|<\delta$ for $i=0,1,\cdots,n$, then*

$$\left\| \prod_{i=0}^{n} I_{t_{i+1}-t_i}p - T(y)p \right\| < \epsilon.$$

PROOF. Let $\epsilon>0$. Using Theorem 1, let δ be a positive number such that if $0=t_0<t_1<\cdots<t_{n+1}=y$ and $|t_{i+1}-t_i|<\delta$ for $i=0,1,2,\cdots,n$, then

$$\limsup_{\delta_0,\delta_1,\cdots,\delta_n\to 0} \left\| \prod_{i=0}^{n} [I-(t_{i+1}-t_i)A_{\delta_i}]^{-1}p - T(y)p \right\| < \epsilon/2.$$

Take $0=t_0<t_1<\cdots<t_{n+1}=y$ where $|t_{i+1}-t_i|<\delta$ for $i=0,1,\cdots,n$. There is a positive number k such that if each of $\delta_0,\delta_1,\cdots,\delta_n$ is less than k, then

$$\left\| \prod_{i=0}^{n} [I-(t_{i+1}-t_i)A_{\delta_i}]^{-1}p - T(y)p \right\| < \epsilon/2.$$

Now $I_{t_{n+1}-t_n}p=\lim_{m\to\infty}[I-(t_{n+1}-t_n)A_{\epsilon_m}]^{-1}p$, and so δ_n can be chosen to be a number in the sequence E which is less than k and such that

$$\| [I-(t_{n+1}-t_n)A_{\delta_n}]^{-1}p - I_{t_{n+1}-t_n}p \| < \epsilon/2(n+1).$$

Since $I_{t_n-t_{n-1}}$ is nonexpansive, it follows that

$$(1)\quad \| I_{t_n-t_{n-1}}[I-(t_{n+1}-t_n)A_{\delta_n}]^{-1}p - I_{t_n-t_{n-1}}I_{t_{n+1}-t_n}p \| < \epsilon/2(n+1).$$

Now

$$\begin{aligned} I_{t_n-t_{n-1}}&[I-(t_{n+1}-t_n)A_{\delta_n}]^{-1}p \\ &= \lim_{m\to\infty}[I-(t_n-t_{n-1})A_{\epsilon_m}]^{-1}[I-(t_{n+1}-t_n)A_{\delta_n}]^{-1}p \end{aligned}$$

and so δ_{n-1} can be chosen to be a number in sequence E which is less than k and such that

$$\tag{2} \begin{aligned} \| [I - (t_n - t_{n-1})A_{\delta_{n-1}}]^{-1}[I - (t_{n+1} - t_n)A_{\delta_n}]^{-1}p \\ - I_{t_n - t_{n-1}}[I - (t_{n+1} - t_n)A_{\delta_n}]^{-1}p\| < \epsilon/2(n+1). \end{aligned}$$

From (1) and (2) it follows that

$$\begin{aligned} \|I_{t_n - t_{n-1}} I_{t_{n+1} - t_n} p - [I - (t_n - t_{n-1})A_{\delta_{n-1}}]^{-1}[I - (t_{n+1} - t_n)A_{\delta_n}]^{-1}p\| \\ < 2\epsilon/2(n+1). \end{aligned}$$

Since $I_{t_{n-1} - t_{n-2}}$ is nonexpansive, it follows that

$$\tag{3} \begin{aligned} \|I_{t_{n-1} - t_{n-2}} I_{t_n - t_{n-1}} I_{t_{n+1} - t_n} p \\ - I_{t_{n-1} - t_{n-2}}[I - (t_n - t_{n-1})A_{\delta_{n-1}}]^{-1}[I - (t_{n+1} - t_n)A_{\delta_n}]^{-1}p\| \\ < 2\epsilon/2(n+1). \end{aligned}$$

Now

$$\begin{aligned} I_{t_{n-1} - t_{n-2}}[I - (t_n - t_{n-1})A_{\delta_{n-1}}]^{-1}[I - (t_{n+1} - t_n)A_{\delta_n}]^{-1}p \\ = \lim_{m \to \infty} [I - (t_{n-1} - t_{n-2})A_{\epsilon_m}]^{-1}[I - (t_n - t_{n-1})A_{\delta_{n-1}}]^{-1} \\ \cdot [I - (t_{n+1} - t_n)A_{\delta_n}]^{-1}p. \end{aligned}$$

Let δ_{n-2} be a number in sequence E which is less than k and such that

$$\tag{4} \begin{aligned} \| [I - (t_{n-1} - t_{n-2})A_{\delta_{n-2}}]^{-1}[I - (t_n - t_{n-1})A_{\delta_{n-1}}]^{-1}[I - (t_{n+1} - t_n)A_{\delta_n}]^{-1}p \\ - I_{t_{n-1} - t_{n-2}}[I - (t_n - t_{n-1})A_{\delta_{n-1}}]^{-1}[I - (t_{n+1} - t_n)A_{\delta_n}]^{-1}p\| \\ < \epsilon/2(n+1). \end{aligned}$$

From (3) and (4) it follows that

$$\begin{aligned} \|I_{t_{n-1} - t_{n-2}} I_{t_n - t_{n-1}} I_{t_{n+1} - t_n} p - [I - (t_{n-1} - t_{n-2})A_{\delta_{n-2}}]^{-1} \\ \cdot [I - (t_n - t_{n-1})A_{\delta_{n-1}}]^{-1}[I - (t_{n+1} - t_n)A_{\delta_n}]^{-1}p\| \\ < 3\epsilon/2(n+1). \end{aligned}$$

Continuing this process of choosing $\delta_n, \delta_{n-1}, \cdots, \delta_0$, it can be seen that

$$\tag{5} \left\| \prod_{i=0}^{n} I_{t_{i+1} - t_i} p - \prod_{i=0}^{n} [I - (t_{i+1} - t_i)A_{\delta_i}]^{-1} p \right\| < (n+1)\epsilon/2(n+1).$$

But each of $\delta_0, \delta_1, \cdots, \delta_n$ is less than k and thus

$$\tag{6} \left\| \prod_{i=0}^{n} [I - (t_{i+1} - t_i)A_{\delta_i}]^{-1} p - T(y)p \right\| < \epsilon/2.$$

From (5) and (6) it follows that

$$\left\| \prod_{i=0}^{n} I_{t_{i+1}-t_i} p - T(y)p \right\| < \epsilon,$$

and the lemma is proved.

It follows from Lemma 10 that $\lim_{n\to\infty} (I_{y/n})^n p = T(y)p$, and thus (iii) of Theorem 2 is proved.

References

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