ON THE L_2 SPACE OF A BANACH LIMIT

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1. **Introduction.** Let Z be the discrete group of integers under addition. The function algebra m of bounded two sided complex sequences is isomorphic to the continuous complex functions $C(\beta Z)$ on the Stone-Čech compactification βZ of Z. We denote by $rca(\beta Z)$ the regular bounded complex Borel measures on βZ. A (two sided) Banach limit is a positive unit norm shift invariant linear functional on m. The mapping $T: Z \to Z$ given by $Tn = n+1$, $n \in Z$, has a continuous extension $T: \beta Z \to \beta Z$, and the Banach limits correspond 1-1 to the probability measures in $rca(\beta Z)$ which are invariant under T. If μ is such an invariant probability we denote by L_μ the corresponding functional on m; both μ and L_μ will be referred to as Banach limits.

The characters of Z are the familiar $\chi_\theta(n) = e^{in\theta}$, $n \in Z$, for each $\theta \in (-\pi, \pi]$. We denote by $\chi_\theta(\xi)$, $\xi \in \beta Z$, the continuous extension of χ_θ to βZ, and we denote by $\mathrm{X} = \{\chi_\theta \in C(\beta Z) : \theta \in (-\pi, \pi]\}$ the set of all such extended characters.

Let $\mu \in rca(\beta Z)$ be a Banach limit, and let $L_2(\beta Z, \mu)$ be the usual L_2 space for the probability measure μ. Let $\mathfrak{M}_\mu$ be the closed linear span of X in $L_2(\beta Z, \mu)$. R. G. Douglas in [1] observes that if μ is not an extreme point of the set of Banach limits, then $\mathfrak{M}_\mu$ is a proper subspace of $L_2(\beta Z, \mu)$, and in [2] he examines the problem of whether $\mathfrak{M}_\mu$ is all of $L_2(\beta Z, \mu)$ when μ is extreme. We show here that for each Banach limit μ, $\mathfrak{M}_\mu$ is a proper subspace of $L_2(\beta Z, \mu)$. The proof is just this: we exhibit functions $f \in C(\beta Z)$ with zero Fourier coefficients ($\int f \bar{\chi}_\theta d\mu = 0$ for all $\theta \in (-\pi, \pi]$) but with $\int |f|^2 d\mu = 1$; that is, unit vectors in $L_2(\beta Z, \mu)$ orthogonal to $\mathfrak{M}_\mu$.

2. **Spectral analysis.** It is shown in [1] that, for each Banach limit μ, the characters X are an orthonormal set in $L_2(\beta Z, \mu)$. For $f \in L_2(\beta Z, \mu)$ we denote by

$$C_\theta(f; \mu) = \int f(\xi)\bar{\chi}_\theta(\xi)\mu(d\xi), \qquad \theta \in (-\pi, \pi], f \in L_2(\beta Z, \mu),$$

the Fourier coefficients of f with respect to the orthonormal basis X of $\mathfrak{M}_\mu$. For $f \in C(\beta Z)$ we have also

$$C_\theta(f; \mu) = L_\mu\{f(n)e^{-in\theta}, n \in Z\}, \qquad \theta \in (-\pi, \pi], f \in C(\beta Z).$$

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For $f \in L_2(\beta Z, \mu)$ we denote by

$$\rho(r; f, \mu) = \int f(\xi + r)\bar{f}(\xi)\mu(d\xi), \qquad r \in Z, f \in L_2(\beta Z, \mu),$$

the covariance of the process $\{f(\xi+r)\}$ in $L_2(\beta Z, \mu)$; by $\xi+r$ we mean $T^r\xi$, $\xi \in \beta Z$. For $f \in C(\beta Z)$ we have also

$$\rho(r; f, \mu) = L_\mu\{f(n + r)\bar{f}(n), n \in Z\}, \qquad r \in Z, f \in C(\beta Z).$$

It is clear that $\rho(r; f, \mu)$, $r \in Z$, is a positive definite function on the group Z. Applying the Raikov theorem, we have

$$\rho(r; f, \mu) = \int_{-\pi}^{\pi} e^{ir\theta} dF(\theta; f, \mu), \qquad r \in Z,$$

where the nonnegative finite measure $dF(\cdot; f, \mu)$ is the spectral power distribution of the stationary stochastic process $\cdots f(\xi-1), f(\xi), f(\xi+1), \cdots$ in $L_2(\beta Z, \mu)$.

The extended characters $\chi_\theta(\xi)$, $\xi \in \beta Z$, have the easily verified property $\chi_\theta(\xi+r) = e^{ir\theta}\chi_\theta(\xi)$, $\xi \in \beta Z, r \in Z$, $\theta \in (-\pi, \pi]$. This is to say, the one-dimensional subspace of $L_2(\beta Z, \mu)$ spanned by χ_θ is invariant under the unitary transformation induced by the μ measure preserving transformation $\xi \rightarrow \xi+1$. For a given $f \in L_2(\beta Z, \mu)$, let us fix $\theta \in (-\pi, \pi]$ and resolve f into its components along the subspace spanned by χ_θ and the orthogonal complement in $L_2(\beta Z, \mu)$. Explicitly, $f = f_1 + f_2$ with $f_1 = C_\theta(f; \mu)\chi_\theta$ and $f_2 = f - C_\theta(f; \mu)\chi_\theta$. It is easily checked that every translate of f_1 is orthogonal to every translate of f_2, and there follows

$$\rho(r; f, \mu) = \rho(r, f_1, \mu) + \rho(r; f_2, \mu), \qquad r \in Z,$$

and hence,

$$dF(\cdot; f, \mu) = dF(\cdot; f_1, \mu) + dF(\cdot; f_2, \mu).$$

But $dF(\cdot; f_1, \mu) = |C_\theta(f; \mu)|^2\delta_\theta(\cdot)$, with δ_θ the unit point measure at θ, so

$$dF(\cdot; f, \mu) = |C_\theta(f, \mu)|^2\delta_\theta(\cdot) + dF(\cdot; f_2, \mu).$$

If $dF(\cdot; f, \mu)$ has no jumps then all Fourier coefficients $C_\theta(f; \mu)$, $\theta \in (-\pi, \pi]$, must be zero; we have proved

THEOREM 1. *Suppose $f \in L_2(\beta Z, \mu)$ is such that the spectral distribution $F(\theta; f, \mu)$, $-\pi < \theta \leq \pi$, is continuous. Then f is orthogonal to $\mathfrak{M}_\mu$ in $L_2(\beta Z, \mu)$.*

In Theorem 2, following, we exhibit functions $f \in C(\beta Z)$ such that

$$\rho(r; f, \mu) = \delta_{r0} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ir\theta} d\theta, \qquad r \in Z.$$

For such a function we have $\rho(0; f, \mu) = \int |f|^2 d\mu = 1$ and $dF(\theta; f, \mu) = d\theta/(2\pi)$; by Theorem 1, f is a unit vector in $L_2(\beta Z, \mu)$ orthogonal to $\mathfrak{M}_\mu$.

For $s = 2, 3, \cdots$, let $\mathcal{C}_s$ be the set of real polynomials $p(x)$ such that the leading term is αx^s with α/π irrational, and let $\mathcal{C} = \bigcup_2^\infty \mathcal{C}_s$.

THEOREM 2. *For each $p \in \mathcal{C}$ let $f_p \in \mathcal{C}(\beta Z)$ be the function determined by $f_p(n) = e^{ip(n)}$, $n \in Z$. Then for each Banach limit μ and each $p \in \mathcal{C}$,*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=M}^{n=M+N-1} f_p(n + r)\bar{f}_p(n) = \delta_{r0} = \rho(r; f_p, \mu),$$

uniformly in $M \in Z$ for each $r \in Z$,

and

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=M}^{n=M+N-1} f_p(n)e^{-in\theta} = 0 = C_\theta(f_p; \mu),$$

uniformly in $M \in Z$ for each $\theta \in (-\pi, \pi]$.

PROOF. The proof is by induction on s. Suppose first that $p(x) = \alpha x^2 + \beta x + \gamma$ with α/π irrational. Then

$$f_p(n + r)\bar{f}_p(n) = \exp(i(\alpha r^2 + \beta r))\exp(2i\alpha rn),$$

and it is straightforward that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=M}^{n=M+N-1} f_p(n + r)\bar{f}_p(n) = \delta_{r0}$$

uniformly in $M \in Z$ for each $r \in Z$.

This is to say, for each $r \in Z$ the sequence $\{f_p(n+r)\bar{f}_p(n),\ n \in Z\}$ is almost convergent to δ_{r0}; by the Theorem of Lorentz (easily extended to the two sided case) we have

$$L_\mu\{f_p(n + r)\bar{f}_p(n),\ n \in Z\} = \delta_{r0} = \rho(r; f, \mu), \qquad r \in Z,$$

for every Banach limit μ. It follows from Theorem 1 that for each $\theta \in (-\pi, \pi]$ the Fourier coefficient

$$C_\theta(f_p; \mu) = L_\mu\{f_p(n)e^{-in\theta},\ n \in Z\}$$

vanishes for every Banach limit μ. Applying the Lorentz theorem again, we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=M}^{n=M+N-1} f_p(n)e^{-in\theta} = 0$$

uniformly in $M \in Z$ for each $\theta \in (-\pi, \pi]$;

in particular, for each $p \in \mathcal{C}_2$ there holds

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=M}^{n=M+N-1} e^{ip(n)} = 0 \quad \text{uniformly in } M \in Z.$$

Suppose the theorem has been proved for $\mathcal{C}_{s-1}$, and let $p(x) = \alpha x^s$ + (lower order) be an element of $\mathcal{C}_s$. Then $p(x+r) - p(x) = \alpha s r x^{s-1}$ + (lower order) is an element of $\mathcal{C}_{s-1}$ when $r \neq 0$, whence

$$\begin{aligned} \lim_{N\to\infty} \frac{1}{N} \sum_{n=M}^{n=M+N-1} & f_p(n+r)\bar{f}_p(n) \\ &= \lim_{N\to\infty} \frac{1}{N} \sum_{n=M}^{n=M+N-1} \exp\left(i[p(n+r) - p(n)]\right) \\ &= \delta_{r0} \quad \text{uniformly in } M \in Z \text{ for each } r \in Z, \end{aligned}$$

using the induction hypothesis. That is, for each $p \in \mathcal{C}_s$ we have $\rho(r; f_p, \mu) = \delta_{r0}$, $r \in Z$, for every Banach limit μ. We argue as before to get $C_\theta(f_p; \mu) = 0$ for all $\theta \in (-\pi, \pi]$, all Banach limits μ, and all $p \in \mathcal{C}_s$, and the theorem follows.

COROLLARY. *For each Banach limit μ, $\mathfrak{M}_\mu$ is a proper subspace of $L_2(\beta Z, \mu)$.*

Weyl [3] has shown that for every $p \in \mathcal{C}$, the fractional parts of the numbers $p(n)/(2\pi)$, $n = 1, 2, 3, \cdots$, are uniformly distributed on $[0, 1)$. That is, for each $\phi(x)$, $-\infty < x < \infty$, which is continuous and with period 2π, there holds

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{n=N} \phi(p(n)) = \frac{1}{2\pi} \int_0^{2\pi} \phi(x)dx.$$

Using Theorem 2, it is easy to strengthen this to the following:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=M}^{n=M+N-1} \phi(p(n)) = \frac{1}{2\pi} \int_0^{2\pi} \phi(x)dx \qquad \text{uniformly in } M \in Z$$

for each such ϕ. That is, the equidistribution of $p(n)/(2\pi)$ mod 1, $n \in Z$, is uniform in the starting position.

Theorem 1 has an obvious extension to any locally compact Abelian group. However, the author is not able to exhibit a function with atomless power spectrum in the general noncompact case.

References

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