ON CONVERGENCE RATES IN THE LAW OF LARGE NUMBERS FOR WEIGHTED SUMS OF INDEPENDENT RANDOM VARIABLES[1]

V. K. ROHATGI

1. Introduction. Let $\{X_n: n \geqq 1\}$ be a sequence of independent random variables and let $\{a_{n,k}: n, k \geqq 1\}$ be a double sequence of real numbers. Write $S_n = \sum_{k=1}^{\infty} a_{n,k} X_k$. Franck and Hanson [1] study, under fairly mild conditions, the rate of convergence of the sequence $\{\Pr(|S_n| > \epsilon)\}$ to zero for the case $t > 1$ (t is a constant to be defined in §2) and remark "For completeness, the case $t \leqq 1$ should be thoroughly treated for the coefficient sequence $a_{n,k} \cdots$." The present work is in the spirit of Theorems 1 to 5 of [1] and is intended to complete these theorems for the case $0 < t \leqq 1$.

2. Preliminaries. Let X_n, $a_{n,k}$ and S_n be as defined in §1. Suppose that C, α, β, ρ and t are constants such that

(1) $\sum_{k=1}^{\infty} |a_{n,k}| \leqq Cn^{\alpha}$,

(2) $\sup_k |a_{n,k}| \leqq Cn^{-\beta}$,

(3) $\sum_{k=1}^{\infty} |a_{n,k}|^t \leqq Cn^{-\rho}$.

As is usual in the literature on this subject we use C as a generic constant.

Note that for $0 < t \leqq 1$

$$\left[\sup_k |a_{n,k}|^t\right] \leqq \sum_k |a_{n,k}|^t$$

so that we may assume

(4) $\beta \geqq (\rho/t)$ and $\beta \geqq -\alpha$.

Also, since

$$\sum_k |a_{n,k}| = \sum_k \{|a_{n,k}|^{1-t} |a_{n,k}|^t\} \leqq \left[\sup_k |a_{n,k}|^{1-t}\right] \sum_k |a_{n,k}|^t,$$

we observe that

(5) $\rho + \alpha + \beta(1-t) \leqq 0$.

We assume further that there exists a constant λ, $0 < \lambda < t \leqq 1$, such that

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(6) $\sum_k |a_{n,k}|^{\lambda} \leqq C$.
This is quite a plausible assumption in view of (3).

In order to facilitate comparison and to draw attention to the formal similarity of our results (for the $0<t\leqq 1$ case) to those obtained in [1] (for the $t>1$ case) we will state the results for all $t>0$. Therefore, in the following we assume, unless stated otherwise, that the parameters satisfy conditions (1) to (6) if $0<t\leqq 1$ and conditions (1) to (4) and the condition

(7) $\rho+\alpha+\beta(1-t)\geqq 0$
whenever $t>1$.

For convenience we write

(8) $F(y)=\sup_k \Pr(|X_k-a_k|>y)$,
where $a_k=0$ if $0<t<1$ (and also for the $t=1$ case of Theorems 1 and 2) and $a_k=EX_k$ if $t>1$ (and also for the $t=1$ case of Theorems 3 and 4).

3. Results.

THEOREM 1. *Let $\beta>0$, $\rho>0$ and, in addition, take $\alpha<\beta$ whenever $t>1$. Then $y^tF(y)\leqq C<\infty$ for all $y>0$ implies*

$$\Pr(|S_n| > \epsilon) \leqq O(n^{-\rho}), \tag{9}$$

for every $\epsilon>0$.

THEOREM 2. *Let $\beta>0$, $\rho>0$ and, in addition, take $\alpha<\beta$ whenever $t>1$. Then, if $y^tF(y)\to 0$ as $y\to\infty$ we have*

$$\Pr(|S_n| > \epsilon) = o(n^{-\rho}), \tag{10}$$

for every $\epsilon>0$.

THEOREM 3. *Let $\beta>0$, $\rho>0$ and, in addition, take $\alpha<\beta$ whenever $t\geqq 1$. Suppose F satisfies*

$$\lim_{y\to\infty} F(y) = 0 \quad \textit{and} \quad \int_0^\infty y^t\,|dF(y)| < \infty, \tag{11}$$

whenever $0<t<1$. Then, if there exists a nondecreasing real valued function G satisfying (11) and such that $G(x)\geqq F(x)$ and

$$\sup_{x\geqq 1}\sup_{y\geqq x}\frac{y^tF(y)}{x^tG(y)} = \gamma < \infty, \tag{12}$$

where γ is a constant, we have

$$\sum_{n=1}^{\infty} n^{\rho-1} \Pr(|S_n| > \epsilon) < \infty \tag{13}$$

for every $\epsilon > 0$.

THEOREM 4. *Let* $\beta > 0$, $\rho > 0$ *and, in addition, take* $\alpha < \beta$ *whenever* $t \geqq 1$. *Then, if F satisfies*

$$\lim_{y \to \infty} F(y) = 0 \quad \textit{and} \quad \int_0^{\infty} y^t \log {}^+ y \, | \, dF(y) \, | < \infty, \tag{14}$$

we have that (13) *holds for every* $\epsilon > 0$.

REMARK 1. We emphasize that the assumption of finite expectations used by Franck and Hanson for the $t > 1$ case is not needed for the $0 < t < 1$ case.

REMARK 2. The $t = 1$ case of all the theorems needs special mention. In Theorems 1 and 2 we do not need the assumption of finite expectations for this case also. For Theorems 3 and 4 the $t = 1$ case is due to Franck and Hanson [1] and uses the assumption of finite expectations but not condition (6).

REMARK 3. Theorem 3 of Franck and Hanson [1] cannot be extended to cover the $0 < t < 1$ case in the sense that conditions (11) are not enough to insure (13). See Theorem 4 above in this connection.

The method of proof for the $0 < t < 1$ case (and the $t = 1$ case of Theorems 1 and 2) parallels that of Franck and Hanson [1]. In fact, if for $\delta > 0$ we define

$$\begin{aligned} Y_{n,k} &= X_k \qquad \text{if } |X_k| \leqq n^{-\delta} |a_{n,k}|^{-1}, \\ &= 0 \qquad \text{otherwise,} \end{aligned}$$

then

$$\Pr(|S_n| > 2\epsilon) \leqq \sum_{k=1}^{\infty} \Pr(|a_{n,k} X_k| > \epsilon) \tag{15}$$

$$+ \sum_{j \neq k} \Pr(|a_{n,k} X_k| > n^{-\delta}) \Pr(|a_{n,j} X_j| > n^{-\delta}) \tag{16}$$

$$+ \Pr\left(\left| \sum_{k=1}^{\infty} a_{n,k} Y_{n,k} \right| > \epsilon \right). \tag{17}$$

The next step consists of examining each of the expressions (15), (16) and (17) separately and showing that the rate at which each approaches zero is appropriate for the theorem in question. Since this does not involve any procedural innovations we omit the details to avoid repetition. However, we wish to indicate how condition (6) is used to show that expression (17) is of the required order. Using Markov's inequality we see that for a positive integer A,

$$\begin{aligned}\Pr\left(\left|\sum_k a_{n,k}Y_{n,k}\right| > \epsilon\right) & \\ \leqq CE\left\{\sum_k |a_{n,k}|\,|Y_{n,k}|\right\}^A & \\ = C\sum{}^*\sum{}^{**}\left(A!\Big/\prod_{k=1}^a m_k\right)\prod_{k=1}^a |a_{n,\phi(k)}|^{m_k}E|Y_{n,\phi(k)}|^{m_k}, &\end{aligned} \tag{18}$$

where the first summation on the right is taken over all integers $a, m_1, \cdots, m_a$ such that $m_k \geqq 1$, $k=1, 2, \cdots, a$, $\sum_k m_k = A$, and distinct sets of integers $\{m_1, \cdots, m_a\}$ appear only once in this sum; the second sum is over all 1-1 mappings ϕ from $\{1, 2, \cdots, a\}$ into positive integers. It is easy to see that the condition $y^tF(y) \leqq C < \infty$ for all $y>0$, which is guaranteed by the hypotheses of each of the four theorems, insures a uniform bound in n and k on $E|Y_{n,k}|^\lambda$ for each fixed λ, $0<\lambda<t\leqq 1$. Choosing λ to satisfy (6) we have

$$\begin{aligned}|a_{n,\phi(k)}|^{m_k}E|Y_{n,\phi(k)}|^{m_k} &\leqq C|a_{n,\phi(k)}|^\lambda\left[\sup_{w,n,k}|a_{n,k}|\,|Y_{n,k}(w)|\right]^{m_k-\lambda} \\ &\leqq C|a_{n,\phi(k)}|^\lambda n^{-\delta(m_k-\lambda)},\end{aligned} \tag{19}$$

where δ is chosen such that $0<\delta<(\rho/2t)$ (this is needed in the treatment of (16)) and the constant C depends on A and λ but not on n and k. Thus (18) is bounded by

$$C\sum{}^*\sum{}^{**}\prod_{k=1}^a |a_{n,\phi(k)}|^\lambda n^{-\delta(m_k-\lambda)} \leqq C\sum{}^* n^{-\delta(A-\lambda a)}\sum{}^{**}\prod_{k=1}^a |a_{n,\phi(k)}|^\lambda.$$

Now note that

$$\sum{}^{**}\prod_{k=1}^a |a_{n,\phi(k)}|^\lambda = \left[\sum_k |a_{n,k}|^\lambda\right]^a \leqq C$$

and it follows that (18) is bounded by

$$C\sum{}^* n^{-\delta(A-\lambda a)}. \tag{20}$$

Now choose A so large that for $\tau>\rho$,

$$A > \tau\delta^{-1}(1-\lambda)^{-1} \qquad (0 < \delta < (\rho/2t)).$$

Then $\delta(A-\lambda a) > \tau$ and each term in the sum $\sum^*$ is $o(n^{-\tau})$. Since $\sum^*$ has only a finite number of terms, the number depending on A but not on n, it follows that (17) is of the appropriate order for all the theorems.

4. Concluding remarks. In conclusion we wish to remark that Theorem 2 may be specialized to yield a part of Theorem 1 of Heyde and Rohatgi [2]. Let $\{X_n: n \geqq 1\}$ be a sequence of independent identically distributed random variables and take $0 < r < t \leqq 1$. If we set $a_{n,k} = n^{-1/r}$ for $k \leqq n$ and $a_{n,k} = 0$ for $k > n$ and take $\alpha = 1 - 1/r$, $\beta = 1/r$, $\rho = (t/r) - 1$ then Theorem 2 yields that $n^{\rho+1} \Pr(|X_1| > n^{1/r}) \to 0$ implies $n^{\rho} \Pr(|\sum_{k=1}^{n} X_k| > n^{1/r}\epsilon) \to 0$ for all $\epsilon > 0$.

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References

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The Catholic University of America