# NOTE ON THE GENERALIZED WHITNEY SUM

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1. **Introduction.** Let $p\colon E \to B$ be a Hurewicz fibration with fibre $F$; let $i\colon F \to E$ be the inclusion. Then $p$ induces a map $r\colon E \cup_i CF \to B$ taking $CF$ to the base-point. If we convert $r$ to a fibre map $\pi\colon E_r \to B$, it is a result of Ganea [2] that the fibre $F_r$ has the weak homotopy type of the join $F * \Omega B$, provided $B$ has the homotopy of a CW-complex.

The purpose of this note is to show that Ganea's fibring may be viewed as the generalized Whitney sum of $E$ with the standard contractible fibring over $B$. In so doing, we generalize a result of Hall [3] and make applicable Hall's theory of Whitney sum fibrings to Ganea's fibration.

2. **Statement of the result.** Let $PB$ be the space of paths in $B$ ending at the base-point; $PB$ is a contractible fibre space over $B$ with fibre $\Omega B$. Then the generalized Whitney sum [3] of $E$ and $PB$, denoted here $E + PB$, is a fibre space over $B$ with fibre $F * \Omega B$. Let $\rho\colon E + PB \to B$ be the projection.

PROPOSITION. *Suppose $B$ has the homotopy type of a* CW-*complex. Then there is a weak homotopy equivalence $v\colon E + PB \to E_r$ such that $\pi v = \rho$.*

Thus, up to weak homotopy type, we may view $E + PB$ as the result of converting $r$ to a fibre map.

Taking $E = PB$, we obtain Hall's result that $PB + PB \to B$ is, up to weak homotopy type, the Barcus-Meyer fibre sequence $\Omega B * \Omega B \to S\Omega B \to B$ [1].

3. **Proof.** Under the hypotheses, Ganea defines a weak homotopy equivalence $w\colon F * \Omega B \to F_r$. It suffices by consideration of the homotopy sequences of the fibre maps $\pi$ and $\rho$, to extend $w$ to $v\colon E + PB \to E_r$ such that $\pi v = \rho$.

Now $E + PB$ is the set of pairs $((1-s)e, s\gamma)$, where $e \in E$, $0 \leqq s \leqq 1$, and $\gamma \in PB$ satisfy $\gamma(0) = p(e)$ and are subject to the identifications $(e, 0\gamma) \sim (e, 0\gamma')$ and $(0e, \gamma) \sim (0e', \gamma)$. $\rho$ is defined by $\rho((1-s)e, s\gamma) = p(e) = \gamma(0)$. $E_r$ consists of triples $(e, t, \omega)$ where $0 \leqq t \leqq 1$, $e \in E$, and $\omega \in B^I$ satisfy $p(e) = \omega(1)$, have $t = 1$ whenever $e \notin i(F)$, and are subject to the identifications $(e, 0, \omega) \sim (e', 0, \omega)$. $\pi$ is defined by $\pi(e, t, \omega) = \omega(0)$.

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Let $\lambda$ be a lifting function [4] for $p: E \to B$; to a point $e \in E$ and a path $\omega \in B^I$ starting at $p(e)$, $\lambda$ associates a path in $E$ covering $\omega$ and starting at $e$. For a path $\omega$, let $\omega_s$ be defined by $\omega_s(t) = \omega(st)$. Then $v$ may be defined by

$$\begin{aligned} v((1-s)e, s\gamma) &= (\lambda(e, \gamma_{2s})(1), 1, \gamma_{2s}) && \text{if } 0 \leqq s \leqq \tfrac{1}{2}, \\ &= (\lambda(e, \gamma)(1), 2-2s, \gamma) && \text{if } \tfrac{1}{2} \leqq s \leqq 1. \end{aligned}$$

That $\pi v = \rho$ is obvious. It remains only to note that the map of $F * \Omega B$ to $F_r$ induced by $v$ is precisely the map $w$ given by Ganea.

## References

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3. I. M. Hall, *The generalized Whitney sum*, Quart. J. Math. Oxford Ser. (2) **16** (1965), 360–384.

4. W. Hurewicz, *On the concept of fiber space*, Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 956–961.

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