A LOCALLY CONNECTED, COMPLETE MOORE SPACE ON WHICH EVERY REAL-VALUED CONTINUOUS FUNCTION IS CONSTANT[1]

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Armentrout [1] has shown that there is a Moore space on which every real-valued continuous function is constant. While this space is connected, it is not known to be complete. In answer to questions raised by F. B. Jones [2] it is shown in this paper that there is a connected, locally-connected, complete, separable Moore space on which every continuous real-valued function is constant. This space is very similar to one whose existence was announced by P. Roy [5].

A topological space is called a Moore space if it satisfies Axiom 0 and the first three parts of Axiom 1 of [4]. A Moore space is said to be complete if it satisfies all of Axiom 1 in [4]. In a Moore space, domains are open sets and the set of regions whose existence is assured by Axiom 1 is a base for the topology.

The construction uses a space which is essentially Armentrout's modification of a Moore space constructed by F. B. Jones [3]. It is redescribed here to make the construction more amenable to geometric intuition. The author is indebted to H. Cook for suggesting this geometric realization of the space Σ.

Let C be a planar disc topologized as follows. If P is on the rim of C, regions containing P shall be interiors of circles lying in C and having only P in common with the rim of C together with P. If P is in C but not on the rim of C, regions containing P shall be interiors of circles lying in C having P in their interior and containing no point of the rim of C. With this topology, C becomes a separable, connected, locally connected, complete Moore space S. The rim of C is a discrete set in S and is the union of two disjoint uncountable sets A and B with the property that any domain in S which contains an uncountable subset of one of them has an uncountable subset of the other in its closure [3]. Let M be a countable dense subset of S containing no point of the rim of C.

Let T be the point set in E^3 obtained by rotating the graph of $y=x^2-1$, $(-1\leqq x\leqq 1)$, in the xy plane about the x-axis. For each positive integer n let C_n be the disc consisting of points in the plane $x=1-1/(n+1)$ which are interior to or on T. For negative integers

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n let C_n be the disc consisting of points on the plane $x = -1-1/(n-1)$ which are interior to or on T. Let C_0 be the disc created by the plane $x=0$ and T.

For each integer n, the disc C_n may be topologized as a copy of C yielding a Moore space S_n containing subsets A_n, B_n and M_n which have the same properties with respect to C_n and S_n that A, B and M have with respect to C and S.

The doubly infinite sequence $\cdots S_{-1}, S_0, S_1, \cdots$ of spaces may be sewn together into a connected space Λ by identifying corresponding points of A_{n-1} and A_n where n is odd and corresponding points of B_{n-1} and B_n where n is even.

Let Σ denote the space obtained from Λ by adjoining the points $(-1, 0, 0)$ and $(1, 0, 0)$ as follows. For each positive integer n the set of all points of all regions containing points of the sets C_j for $j \geqq n$ together with the point $(1, 0, 0)$ shall be a region containing $(1, 0, 0)$. Regions containing $(-1, 0, 0)$ are defined similarly.

An extension of Jones' proof in [3] that his space is a Moore space which is not completely regular shows that every real-valued continuous function on Σ has the same value at $(-1, 0, 0)$ as it has at $(1, 0, 0)$. These two points will be called the "ends" of Σ.

Let $x_1, x_2, x_3, \cdots$ be a numbering of the points of M. This induces a numbering $x_{1n}, x_{2n}, x_{3n}, \cdots$ of points of M_n for each integer n. Join each $x_{j,n-1}$ to the point $x_{j,n}$ with a copy $\Sigma_{j,n}$ of the space Σ by identifying one end of $\Sigma_{j,n}$ with $x_{j,n-1}$ and the other end with $x_{j,n}$. This may be visualized as connecting the points $x_{j,n-1}$ and $x_{j,n}$ by a copy T_{jn} of T lying except for its ends between C_{n-1} and C_n and containing a copy Σ_{jn} of Σ as T contains Σ. Further, these T_{jn}'s may be placed so that no two intersect except at a common end point and they are all subsets of the interior of T. This construction yields a Moore space Σ_1 whose points are the points of all the spaces Σ_{jn} subject to the above-mentioned identifications and whose topology is described as follows.

Let $G_1, G_2, \cdots$ be a sequence of collections of regions in Σ satisfying Axiom 1 of [4]. For each $\alpha = (j, n)$ the space Σ_α is a copy of Σ and has a corresponding sequence $G_{\alpha 1}, G_{\alpha 2}, G_{\alpha 3}, \cdots$ which satisfies Axiom 1. There is a sequence $G_{11}, G_{12}, G_{13}, \cdots$ of collections of connected subsets of Σ_1 such that if "region of Σ_1" be defined as an element of G_{11}, then Σ_1 is a complete Moore space. The regions of each G_{1n} may be thought of as being of two types. First, every region of $G_{\alpha,n}$ which does not contain an end of Σ_α is a region of G_{1n}. Second, regions g of G_n which are augmented by the addition of points of the space $\{\Sigma_\alpha\}$ as follows. If g contains both end points of some Σ_α then the augmentation of g contains all of Σ_α. If g contains only one end point of a Σ_α then g is augmented by regions of $G_{\alpha n}$ which contain that end of Σ_α.

The space Σ_1 contains Σ as a subspace and has the property that every real-valued continuous function on Σ_1 is constant on Σ. To see this, consider the sequence of points $\cdots x_{j,-1}, x_{j,0}, x_{j,1}, \cdots$ for some fixed j. Since these points are ends of the spaces Σ_{jn}, $n=0, \pm 1, \pm 2, \cdots$ and have the ends of Σ as limit points, any real-valued continuous function over Σ_1 is constant over this set. Since the set of all ends of the space Σ_{ij}, $i, j=0, \pm 1, \pm 2, \cdots$ is dense in the space Σ the space Σ_1 has the property stated above.

By placing copies of T inside each T_{ij} in the same way that T_{ij}'s were placed in T, a Moore space Σ_2 may be constructed which contains Σ_1 as a subspace so that every real-valued continuous function Σ_2 is constant on Σ_1. This process may be continued, yielding an infinite sequence $\Sigma_1, \Sigma_2, \Sigma_3, \cdots$ of Moore spaces such that for each positive integer j, Σ_{j+1} contains Σ_j as a subspace and every real-valued continuous function on Σ_{j+1} is constant on Σ_j.

In each space Σ_j, $j>1$, there is a sequence $G_{j1}, G_{j2}, \cdots$ of collections of regions obtained from regions of the space Σ_{j-1} in the same way that regions for Σ_1 were obtained from regions in Σ. Note that if P is a point of Σ_j not in Σ_{j-1} then P is not an end point of a copy of Σ in Σ_j and for some integer k, regions of G_{jk} which contains P are "between" some two discs in the copy of Σ which contains P. Further, no future augmentation of such regions will include points not between those two discs.

Let L be the set of all points in spaces in the sequence $\{\Sigma_i\}$, $i=1, 2, 3, \cdots$ and K be the set of points in E^3 so that $P\in K$ if and only if there is an infinite nested sequence of copies of T used in the construction of the spaces $\{\Sigma_j\}$ so that P is interior to all the copies of T in this sequence. The set $K\cup L$ may be topologized to form a space Σ_∞ so that each Σ_i is a subspace of Σ_∞ and Σ_∞ satisfies the requirements listed in the title.

To do this, we first indicate what sets will be regions containing points of K. If $P\in K$, then in the construction of Σ_{n+1} there was only one copy of T inserted between discs of Σ_n which contains P in its interior. The set of all points of $K\cup L$ interior to this copy of T will be a region containing P. For points x in L, we observe that there is a smallest integer j such that x is a point of Σ_j. Let g be a region of Σ_j containing P. There is a sequence $g_1, g_2, g_3, \cdots$ such that for each m, g_{m+1} is a region of Σ_{m+1} which was obtained from g_m in Σ_m by an augmentation process as described above. Further, g_1 is obtained from g in this fashion. We create a region of Σ_∞ containing P by taking the union of the sequence $g_1, g_2, g_3, \cdots$ together with all points of K which are interior to some copy of T used in the construction of a copy of Σ which is a subset of some g_i. From this collection of regions it is possible to create a sequence of collections $H_1, H_2, H_3, \cdots$ of regions of Σ_∞ which satisfies all of Axiom 1 of [4] and further having the property that each region is a connected set which lies between some two discs of each Σ_j which it does not intersect.

The completeness of the space Σ_∞, i.e. the fourth part of the Axiom 1 of [4] requires the following statement to be true. If $M_1, M_2, M_3, \cdots$ is a decreasing sequence of closed sets in Σ_∞ such that for each positive integer n, there is a region h_n of H_n such that $M_n \subset \mathrm{cl}\, h_n$, then there is a point common to all sets of the sequence $\{M_n\}$.

Since each Σ_j is a complete Moore space in the relative topology created by the sequence $H_1, H_2, H_3, \cdots$ any such sequence $\{M_n\}$ which fails to have a common point must eventually fail to intersect any given Σ_j. This then implies that for some i, M_i is a subset of a copy of T used in constructing a copy of Σ to build Σ_{j+1} from Σ_j. Thus one can follow the sequence $\{M_n\}$ with a nested sequence of copies of T and conclude that some point of K is common to the sequence $\{M_i\}$.

Thus the space Σ_∞ satisfies the requirements listed in the title.

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