ON A COHOMOLOGY THEORY FOR PAIRS OF GROUPS

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Let H be a subgroup of a group G, and let A be a left G-module. Consider the abelian group

$$X(G, H, A) = \{f: G \to A \mid f(xy) = xf(y) + f(x), f_{|H} = 0\}$$

of crossed homomorphisms from G to A vanishing on H. Clearly this is a left-exact functor in the category ${}_G\mathfrak{M}$ of left G-modules. The nth right derived functor of $X(G, H, -)$ in ${}_G\mathfrak{M}$ is denoted by $H^{n+1}(G, H, -)$. The group $H^n(G, H, A)$, $A \in {}_G\mathfrak{M}$, is called the nth cohomology group of the pair (G, H) with coefficients in A. These groups were first described and studied by M. Auslander in [**1**], who also found the sequence of Proposition 1.2.

In this note we prove an excision property for the functors $H^n(G, H, -)$, Theorem 2.2, and we find a direct sum decomposition of them under suitable conditions, Propositions 2.3 and 2.5. From this one deduces by standard methods a Mayer-Vietoris type sequence for the cohomology of groups, Proposition 2.6.

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1. Let $H \subset G$ be groups, and let $A \in {}_G\mathfrak{M}$. Then $A^* = \mathrm{Hom}_H(ZG, A)$ is a left G-module in the obvious way (ZG denotes the integral group ring, and $\mathrm{Hom}_H(-, -) \equiv \mathrm{Hom}_{ZH}(-, -)$). Let Z be the group of integers with G-structure defined by $xn = n$, $x \in G$, $n \in Z$. Then one has the natural isomorphisms

$$\begin{aligned}\mathrm{Hom}_G(Z, A^*) &= \mathrm{Hom}_G(Z, \mathrm{Hom}_H(ZG, A)) \\ &\approx \mathrm{Hom}_H(ZG \otimes_G Z, A) \approx \mathrm{Hom}_H(Z, A).\end{aligned}$$

Hence, since every G-injective module is H-injective, cf. [**4**, p. 31, Proposition 6.2a], one has

$$H^n(G, A^*) \approx H^n(H, A)$$

for $A \in {}_G\mathfrak{M}$. (A^* is exact in ${}_G\mathfrak{M}$ and preserves injectives.)

Now let $\gamma: A \to A^*$ be the G-monomorphism defined by $\gamma(a)x = xa$, $a \in A$, $x \in G$. Let $\Gamma = \operatorname{coker} \gamma$.

LEMMA 1.1. $\mathrm{Hom}_G(Z, \Gamma) \approx X(G, H, A)$ *as functors in* ${}_G\mathfrak{M}$.

PROOF. Notice that $\Gamma = A^*/\mathrm{Im}\, \gamma$. We make the identifications

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$$A^* = \mathrm{Hom}_H(ZG, A) = \{f: G \to A \mid f(xy) = xf(y),\ x \in H,\ y \in G\},$$
$$\mathrm{Im}\,\gamma = \mathrm{Hom}_G(ZG, A) = \{g: G \to A \mid g(xy) = xg(y),\ x, y \in G\}.$$

Let $f \in A^*$ and assume $f + \mathrm{Im}\,\gamma \in \Gamma^G = \mathrm{Hom}_G(Z, \Gamma)$. Then $zf - f \in \mathrm{Im}\,\gamma$, $\forall z \in G$; so $(zf-f)(xy) = x(zf-f)(y) = xf(yz) - xf(y)$, and also $(zf-f)(xy) = f(xyz) - f(xy)$, $\forall x, y, z \in G$. Let $z = y^{-1}$; then $f(xy) = xf(y) + f(x) - xf(1)$. Define $f_c \in \mathrm{Im}\,\gamma$ by $f_c(x) = xf(1)$; then $(f - f_c) + \mathrm{Im}\,\gamma = f + \mathrm{Im}\,\gamma$, and $f - f_c \in X(G, H, A)$. It is easily checked now that the map $f + \mathrm{Im}\,\gamma \mapsto f - f_c$ is a natural isomorphism from $\mathrm{Hom}_G(Z, \Gamma)$ to $X(G, H, A)$.

PROPOSITION 1.2. *Let $H \subset G$ be groups, and let $A \in {}_G\mathfrak{M}$. Then there exists a long exact sequence*

$$0 \to A^G \xrightarrow{i} A^H \xrightarrow{\delta} H^1(G, H, A) \xrightarrow{j} H^1(G, A) \xrightarrow{i} H^1(H, A) \xrightarrow{\delta} H^2(G, H, A) \xrightarrow{j} \cdots$$

where the i's are restriction maps induced by the inclusion $H \to G$.

PROOF. Apply $\mathrm{Ext}_G(Z, -)$ to the short exact sequence $0 \to A \to A^* \to \Gamma \to 0$. ($\mathrm{Ext}^n_G(Z, \Gamma(A)) \approx H^{n+1}(G, H, A)$ by Lemma 1.1, since $\Gamma(A)$ is exact and $\mathrm{Ext}_G(Z, \Gamma(A))$ is effaceable in ${}_G\mathfrak{M}$.)

COROLLARY 1.3. *Let* 1 *denote the group with one element. Then* $H^n(G, 1, A) \approx H^n(G, A)$, $n \geqq 2$, $A \in {}_G\mathfrak{M}$.

2. Let $H \subset G$, $L \subset K$ be groups. Let $\varphi: K \to G$ be a group homomorphism with $\varphi L \subset H$. If $A \in {}_G\mathfrak{M}$, denote by ΦA the corresponding K-module structure in A induced by φ, $x \cdot a = \varphi(x)a$, $x \in K$, $a \in A$. Then φ induces a natural homomorphism $\varphi^1: X(G, H, A) \to X(K, L, \Phi A)$ defined by $(\varphi^1 f)x = f(\varphi x)$, which in turn induces mappings $\varphi^n: H^n(G, H, A) \to H^n(K, L, A)$. If φ is the inclusion we will denote ΦA by A again.

LEMMA 2.1. *Let $H \subset K \subset G$ be groups. Then $\{H^n(K, H, -) \mid n \geqq 1\}$ is a universal sequence of connected functors in ${}_G\mathfrak{M}$ ("∂-foncteur universel" in the terminology of* [6]).

PROOF. The sequence is certainly exact. So, it suffices to show that it is effaceable (see [6, Proposition 2.2.1]). If A is a G-injective module, then it is K-injective, since ZG is K-free (see [4, p. 31, Proposition 6.2a]). Thus $H^n(K, H, A) = 0$ if $n > 1$.

Now let H and K be groups with a common subgroup L, and denote by $H *_L K$ the amalgamated product of H and K with amalgamated subgroup L (i.e., the pushout of $L \to H$ and $L \to K$ in the category of groups), cf. [7, p. 312].

THEOREM 2.2 (EXCISION AXIOM). *Let L be a common subgroup of groups H and K and let $G=H *_L K$. Then the morphisms of functors in ${}_G\mathfrak{M}$,*

$$\varphi^n: H^n(G, H, -) \to H^n(K, L, -), \qquad n \geqq 1,$$

induced by the inclusion $\varphi: (K, L) \to (G, H)$, are isomorphisms.

PROOF. By Lemma 2.1, it suffices to show that φ^1 is an isomorphism. Let $A \in {}_G\mathfrak{M}$; if $f \in X(G, H, A)$ and $k \in K$, then, by definition, $(\varphi^1 f)k = fk$. Consider the map $\psi: X(K, L, A) \to X(G, H, A)$ defined in the following manner. Let $g \in X(K, L, A)$ and $x \in G$; suppose $a_1a_2 \cdots a_n$ is a representative word of x (a_i belongs either to H or to K, $i=1, 2, \cdots, n$). Then set

$$(\psi g)x = g'(a_1) + a_1 g'(a_2) + \cdots + a_1 a_2 \cdots a_{n-1} g'(a_n)$$

where $g'(a_i) = g(a_i)$ if $a_i \in K$, and $g'(a_i) = 0$ if $a_i \in H$.

It is easily proved that ψg is a well-defined crossed homomorphism of G to A vanishing on H, i.e. $\psi g \in X(G, H, A)$. Moreover, ψ is a homomorphism.

On the other hand it is plain that $\varphi^1\psi = \text{id.}$ on $X(K, L, A)$; also, if $a_1a_2 \cdots a_n$ is a representative word of $x \in G$, and $f \in X(G, H, A)$, then

$$\begin{aligned}(\psi\varphi^1)(f)(x) &= (\varphi^1 f)'(a_1) + a_1(\varphi^1 f)'(a_2) + \cdots + a_1a_2 \cdots a_{n-1}(\varphi^1 f)'(a_n)\\ &= f(a_1) + a_1 f(a_2) + \cdots + a_1 a_2 \cdots a_{n-1} f(a_n)\\ &= f(a_1 a_2 \cdots a_n) = f(x),\end{aligned}$$

i.e. $\psi\varphi^1 = \text{id.}$ on $X(G, H, A)$. Thus φ^1 is an isomorphism.

PROPOSITION 2.3. *Let $G = H *_L K$ where L is a common subgroup of groups H and K. Then*

$$H^n(G, L, A) \approx H^n(H, L, A) \oplus H^n(K, L, A),$$

for $n \geqq 1$ and $A \in {}_G\mathfrak{M}$, where the canonical projections are induced by the inclusions $(H, L) \to (G, L)$ and $(K, L) \to (G, L)$.

PROOF. By Lemma 2.1 it suffices to show that the result holds on dimension 1. If $f \in X(G, L, A)$, define $\varphi f = (f_1, f_2)$, where $f_1 \in X(H, L, A)$ and $f_2 \in X(K, L, A)$ are the restrictions of f to H and K respectively. Conversely, given $g_1 \in X(H, L, A)$ and $g_2 \in X(K, L, A)$, define $\psi(g_1, g_2) = g: G \to A$ as follows: if $a_1a_2 \cdots a_n$ is a representative word of $x \in G$, put $g(x) = g'(a_1) + a_1 g'(a_2) + \cdots + a_1 a_2 \cdots a_{n-1} g'(a_n)$, where $g'(a_i)$ is $g_1(a_i)$ or $g_2(a_i)$ depending on whether a_i is in H or K respectively (notice that $g_{1|L} = g_{2|L} = 0$). Then g is a well-defined crossed homomorphism of G to A vanishing on L, i.e. $g \in X(G, L, A)$. Hence φ and ψ are inverse isomorphisms as desired.

REMARK. Proposition 2.3 has also been proved independently by M. Barr and J. Beck; see [2].

COROLLARY 2.4 (LYNDON [8]; BARR AND RINEHART [3]). *Let $G = H * K$ (free product of groups H and K) and let $A \in {}_G\mathfrak{M}$. Then $H^n(G, A) = H^n(H, A) \oplus H^n(K, A)$ if $n \geqq 2$.*

PROOF. Put $L = 1$ in Proposition 2.3 and apply Corollary 1.3.

We now prove a converse to Proposition 2.3. Notice first that given a group T and an abelian group B, a T-module structure on B is nothing but a group homomorphism $T \to \mathrm{Aut}(B)$, where $\mathrm{Aut}(B)$ is the group of automorphisms of B.

PROPOSITION 2.5. *Let H and K be subgroups of a group G, and let $L = H \cap K$. Assume that for every abelian group A and every pair $\varphi_1: H \to \mathrm{Aut}(A)$, $\varphi_2: K \to \mathrm{Aut}(A)$ of group homomorphisms that coincide on L there is a group homomorphism $\varphi: G \to \mathrm{Aut}(A)$ extending φ_1 and φ_2. Suppose, moreover, that the isomorphisms of Proposition 2.3 hold. Then $G = H *_L K$.*

PROOF. In particular

$$X(G, L, A) \approx X(H, L, A) \oplus X(K, L, A)$$

for $A \in {}_G\mathfrak{M}$, i.e. every pair $f_1: H \to A$, $f_2: K \to A$ of crossed homomorphisms vanishing on L extends uniquely to a crossed homomorphism $f: G \to A$. Let G_1 be the subgroup of G generated by H and K; we first show that $G = G_1$. Assume $G \neq G_1$. If $x \in G$, let $\bar{x}$ denote the corresponding left coset of G_1 in G. Let $I = I(G/G_1)$ be the free abelian group generated by $\{\bar{x} - 1 \mid 1 \neq x \in G\}$, and let G act on I by $y(\bar{x} - 1) = ((yx)^{-} - 1) - (\bar{y} - 1)$, $x, y \in G$. Then $I \in {}_G\mathfrak{M}$. Let $f_1: H \to I$ and $f_2: K \to I$ be the zero crossed homomorphisms; these extend to the zero crossed homomorphism $G \to I$. On the other hand $f: G \to I$, defined by $fx = \bar{x} - 1$, $x \in G$, is plainly a nonzero crossed homomorphism extending f_1 and f_2, contradicting the hypothesis. Hence $G = G_1$.

We will see now that

$$\begin{array}{ccc} L & \to & H \\ \downarrow & & \downarrow \\ K & \to & G \end{array}$$

is a pushout diagram (all maps are inclusions), i.e. $G = H *_L K$. Suppose P is a group and let $\varphi_1: H \to P$ and $\varphi_2: K \to P$ be group homomorphisms that coincide on L. Denote by $F(P)$ the free abelian group on the set P, and consider a standard embedding $P \to \mathrm{Aut}(F(P))$. Then by assumption φ_1 and φ_2 extend to a group homomorphism $\varphi: G \to \mathrm{Aut}(F(P))$. However, since G is generated by H and K, φ must be unique and into P.

Finally, we state the following proposition whose proof is formally as in Theorem 15.3(c), p. 43 of [5], and which is therefore omitted.

PROPOSITION 2.6 (A MAYER-VIETORIS SEQUENCE). *Let L, H, K, G and A be as in Theorem 2.2. Then the sequence*

$$\cdots \to H^{q-1}(L, A) \xrightarrow{\Delta} H^q(G, A)$$

$$\xrightarrow{\phi} H^q(H, A) \oplus H^q(K, A) \xrightarrow{\Psi} H^q(L, A) \to \cdots$$

where $\Delta = H^{q-1}(L, A) \xrightarrow{\delta} H^q(K, L, A) \xrightarrow{(\varphi^q)^{-1}} H^q(G, H, A) \to H^q(G, A)$ with δ and j as in Proposition 1.2, and φ^q as in Theorem 2.2; ϕ is the direct sum of the maps induced in cohomology by the inclusions $H \to G$ and $K \to G$; $\Psi(v_1, v_2) = h_1^q v_1 - h_2^q v_2$, where h_1^q and h_2^q are maps induced in cohomology by the inclusions $h_1: L \to H$ and $h_2: L \to K$ respectively, $v_1 \in H^q(H, A)$, $v_2 \in H^q(K, A)$.

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