CONCERNING SEMICONNECTED MAPS

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Introduction. Professor John Jones, Jr., [3], introduces a semiconnected map \( f: X \to Y \) as one in which \( f^{-1} \) preserves closed connected subsets of \( Y \), and gives conditions under which a semiconnected map is continuous or is a homeomorphism. Theorem 1 of that paper is generalized here, and comparisons are made between semiconnected maps and other noncontinuous maps.

Among the several other well-known types of noncontinuous maps only the connected map and the connectivity map will be considered. A connected map \( f: X \to Y \) is one which preserves connected subsets of \( X \) and a connectivity map \( f: A \to F \) is one for which the induced graph map, \( g: X \to X \times Y \) defined by \( g(x) = (x, f(x)) \) for each \( x \in X \), is connected. It is easy to see that if \( f: X \to Y \) is continuous, then \( f \) is a connectivity map, and if a connectivity map, then also connected. Examples showing the reverse implications are not always valid may be found in [2]. The example \( f(x) = x^2 \) from the reals into the reals (usual topology in both cases) shows that continuous maps, hence connected and connectivity maps, need not be semiconnected. Furthermore, \( f(x) = x \) from the reals (usual topology) to the reals (discrete topology) is semiconnected but not connected, hence not a connectivity nor a continuous map.

Throughout, \( \overline{cl}(A) \) denotes the closure of the set \( A \).

Results. Theorem 1 generalizes Theorem 1 of [3].

Theorem 1. If \( f: X \to Y \) is semiconnected and onto the semi-locally-connected space \( Y \), then \( f \) is continuous.

Proof. Let \( P \subset F \) be open. It will be shown that \( f^{-1}(P) \) is open in \( X \). For each point \( b \in B \) there exists an open set \( V_b \subset B \) such that \( Y - V_b \) consists of a finite number of components \( C_1, C_2, \ldots, C_k \). Each \( C_j \) is closed and connected; hence \( f^{-1}(C_j) \) is closed and connected since \( f \) is semiconnected. Thus \( \bigcup_{j=1}^k f^{-1}(C_j) \) is closed and contains no point of \( f^{-1}(V_b) \) so that \( X - \bigcup_{j=1}^k f^{-1}(C_j) = R_b \) is an open set in \( X \) having the property that \( f(R_b) = V_b \). Consequently \( \bigcup_{b \in B} R_b \) is open in \( X \) and furthermore \( f^{-1}(B) = \bigcup_{b \in B} R_b \).

Theorem 2. Let \( f: X \to Y \) be a closed map where \( f^{-1}(y) \) is connected for each \( y \in Y \). Then if \( M \subset Y \) is connected, \( f^{-1}(M) \) is connected.

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Proof. Considering \( M \subset Y \) nondegenerate, suppose \( f^{-1}(M) = H \cup K \), separated. Then \( f(H) \cup f(K) = M \) and one of the sets, say \( f(H) \), has a limit point \( y_0 \) of the other, \( f(K) \) in this case. Since \( f^{-1}(y) \) is connected for each \( y \in Y \), \( f^{-1}(y_0) \subset H \) and furthermore \( f(H) \cap f(K) = \emptyset \). Consequently, because \( \text{cl}(K) \cap H = \emptyset \), \( y_0 \in f(\text{cl}(K)) \) which contradicts \( f \) being closed. The conclusion that \( f^{-1}(M) \) is connected follows.

Corollary 1. Let \( f : X \to Y \) be a closed semiconnected map where \( Y \) is \( T_1 \). Then if \( M \subset Y \) is any connected set, \( f^{-1}(M) \) is connected.

Corollary 2. Let \( f : X \to Y \) be a closed connected map where \( f^{-1}(y) \) is connected for each \( y \in Y \) and \( Y \) is \( T_1 \). Then \( f \) is semiconnected.

Proof. For any closed connected \( M \subset Y \), \( f^{-1}(M) \) is connected by Theorem 2. By [4] \( f^{-1}(M) \) is also closed and hence \( f \) is semiconnected.

Corollary 3. If \( f : X \to Y \) is a closed continuous map where \( f^{-1}(y) \) is connected for each \( y \in Y \), and \( Y \) is \( T_1 \), then \( f \) is semiconnected.

Theorem 3. Let \( f : X \to Y \) be continuous where \( f^{-1}(y) \) is connected for each \( y \in Y \), \( X \) is countably compact first countable and \( Y \) is \( T_1 \) first countable. Then \( f \) is semiconnected.

Proof. Let \( M \subset Y \) be closed and connected. Continuity of \( f \) insures \( f^{-1}(M) \) closed. It will now be shown that \( f^{-1}(M) \) is connected from which the conclusion that \( f \) is semiconnected follows.

Suppose \( f^{-1}(M) = H \cup K \), separated. Then \( f(H) \cup f(K) = M \) and one of these sets, say \( f(H) \), has a limit point \( y_0 \) of the other, \( f(K) \) in this instance. There exists a sequence of distinct points \( y_n \in f(K) \) such that \( y_n \to y_0 \) where \( f^{-1}(y_0) \subset H \) and \( f^{-1}(y_n) \subset K \) for each \( n \). Extracting \( x_n \in K \cap f^{-1}(y_n) \) for each \( n \), the set \( \{x_n\} \) has a limit point \( x_0 \in H \) since \( \text{cl}(K) \cap H = \emptyset \). Thus \( f(x_0) \neq y_0 \). Since \( X \) is first countable, there is a subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) such that \( x_{n_k} \to x_0 \). But \( f(x_{n_k}) \to y_0 \neq f(x_0) \) contradicting continuity of \( f \) [1, Theorem 3.15, p. 102]. Thus \( f^{-1}(M) \) is connected.

References


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