GENERALIZATION OF A THEOREM OF PÓSA

Hudson V. Kronk

A Hamiltonian cycle in a graph $G$ is a cycle containing all the points of $G$, and a graph with a Hamiltonian cycle is called Hamiltonian. In [3], Pósa proved the following interesting and important theorem.

**Theorem of Pósa.** Let $G$ be a graph on $p \geq 3$ points such that for every integer $i$ with $1 \leq i < p/2$, the number of points of degree not exceeding $i$ is less than $i$. Then $G$ is Hamiltonian.

A graph $G$ on $p \geq 3$ points is said to be $k$-path Hamiltonian if every path of length not exceeding $k$, $0 \leq k \leq p - 2$, is contained in a Hamiltonian cycle of $G$. The $0$-path Hamiltonian graphs are then the Hamiltonian graphs. The object of this note is to generalize Pósa's theorem to $k$-path Hamiltonian graphs. The proof is an extension of that used in the proof of Theorem 1 of [2].

**Theorem.** Let $G$ be a graph with $p \geq 3$ points, and let $0 \leq k \leq p - 3$. If for every integer $i$ with $k + 1 \leq i \leq (p + k)/2$, the number of points of degree not exceeding $i$ is less than $i - k$, then $G$ is $k$-path Hamiltonian.

**Proof.** Assume that $G$ satisfies the hypothesis of the theorem but contains a path $P$ of length not exceeding $k$ which is not contained in a Hamiltonian cycle. We may assume that $G$ becomes $k$-path Hamiltonian whenever any new line is added to $G$. For if $G$ did not originally have this property we could add suitable lines until it did and the resulting graph would still satisfy the hypothesis of the theorem.

Let $v_1$ and $v_p$ be two nonadjacent points of $G$ such that (1) $\rho(v_1) \leq \rho(v_p)$, where $\rho(v)$ denotes the degree of the point $v$, and (2) $\rho(v_1) + \rho(v_p)$ is as large as possible. If we add the line $v_1v_p$ to $G$ we obtain a $k$-path Hamiltonian graph $G'$. Let $C$ be a Hamiltonian cycle of $G'$ which contains the path $P$. Clearly $C$ must include the line $v_1v_p$ and hence $v_1$ and $v_p$ are the endpoints of a spanning path $Q = (v_1, v_2, \ldots, v_p)$ in $G$ which contains the path $P$. If $v_i$, $2 \leq i < p$, is adjacent to $v_1$ and if $v_{i-1}v_i$ is not in $P$, then $v_i-v_p$ is not in $G$. For otherwise, $(v_1, v_i, v_{i+1}, \ldots, v_p, v_{i-1}, v_{i-2}, \ldots, v_1)$ would be a Hamiltonian cycle of $G$ containing $P$. Since at most $k$ lines of $Q$ belong to $P$, it follows that there are at least $\rho(v_i) - k$ points in $G$ which are nonadjacent to $v_p$. Therefore, $\rho(v_1) \leq \rho(v_p) \leq (p - 1) - (\rho(v_1) - k)$ so that $\rho(v_1) \leq (p + k - 1)/2$. Furthermore, whenever $v_i$ is adjacent to $v_1$ and $v_{i-1}v_i$...

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is not in $P$, $(v_{i-1}, v_{i-2}, \ldots, v_1, v_i, v_{i+1}, \ldots, v_p)$ is a spanning path in $G$ containing $P$. By the manner in which $v_1$ and $v_p$ were chosen, it follows that $\rho(v_{i-1}) \leq \rho(v_1)$. Thus, there are at least $\rho(v_1) - k$ points having degree not exceeding $\rho(v_1)$. However, $k + 1 \leq \rho(v_1) \leq (p + k - 1)/2 < (p + k)/2$ so that by assumption there are less than $\rho(v_1) - k$ points having degree not exceeding $\rho(v_1)$. Having been led to a contradiction, we conclude that the theorem is true.

**Corollary.** If $G$ is a graph with $p \geq 3$ points such that each point has degree at least $(p + k)/2$, $0 \leq k \leq p - 2$, then $G$ is $k$-path hamiltonian.

It is not difficult to construct examples that show that the theorem and its corollary are each, in a sense, best possible. However, the problem of finding conditions which are both necessary and sufficient for a graph to be $k$-path hamiltonian remains unsolved and appears to be extremely difficult. It was, however, shown in [1] that a graph $G$ is $(p - 2)$-path hamiltonian if and only if $G$ is (1) the cycle $C_p$, (2) the complete graph $K_p$, or (3) the complete bipartite graph $K(p/2, p/2)$, where (3) is possible only if $p$ is even.

**References**


Western Michigan University and
State University of New York at Binghamton