A HISTORICAL NOTE ON COMPLEX QUADRATIC FIELDS WITH CLASS-NUMBER ONE

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Let d be the discriminant of a quadratic field, $h(d)$ the class-number of the field and χ_d the character (mod $|d|$) given by $\chi_d(n)=(d/n)$ (Kronecker symbol). If we restrict d to values <-8 with $h(d)=1$ then $|d|$ is a prime $\equiv 3$ (mod 8) and it has recently been proved [**1**], [**4**] that $d \geqq -163$. The purpose of this note is to show that Gelfond and Linnik missed an opportunity to prove this result in 1949.

Let k be the discriminant of a real quadratic field and ϵ_0 the fundamental unit of $Q(\sqrt{k})$. In the sequel, we assume $d<-8$ and $h(d)=1$. The following expansion was known for prime k by Heilbronn and Linfoot [**3**] and has recently been established [**5**] for all k such that $(k, d)=1$,

$$
\begin{aligned}
&L(s, \chi_k)L(s, \chi_{kd}) \\
&\quad = \zeta(2s) \prod_{p|k} (1 - p^{-2s}) + \left(\frac{k\sqrt{|d|}}{2}\right)^{1-2s} \frac{\sqrt{\pi}\,\Gamma(s-\frac{1}{2})}{\Gamma(s)} \\
&\qquad \cdot \zeta(2s-1) \prod_{p|k} (1 - p^{2s-2}) + R(s)
\end{aligned} \tag{1}
$$

where $R(1)=O(|d|^{-1/2}\exp[-\pi\sqrt{|d|}/k])$ and the letter p denotes that the products are to be extended over all prime divisors of k. Here and later O refers to $|d|\to\infty$ and k is fixed. By means of Dirichlet's formula, we have at $s=1$,

$$
\begin{aligned}
h(kd)h(k)\log \epsilon_0 &= \frac{\pi k\sqrt{|d|}}{12} \prod_{p|k} (1 - p^{-2}) \\
&\quad + \begin{Bmatrix} 0 \text{ if } k \text{ has two distinct prime factors} \\ -\log p \text{ if } k \text{ is a power of } p \end{Bmatrix} \\
&\quad + O(e^{-\pi\sqrt{|d|}/k}).
\end{aligned} \tag{2}
$$

As a corollary of (2),

$$h(kd) = O(\sqrt{|d|}). \tag{3}$$

By using $k=5$ and 13 in (2), Gelfond and Linnik [**2**, pp. 38–40] arrived at

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$$35h(5d)\log\left(\frac{1+\sqrt{5}}{2}\right)-13h(13d)\log\left(\frac{3+\sqrt{13}}{2}\right)+\log\left(\frac{5^{35}}{13^{13}}\right) = O(\exp[-\pi\sqrt{|d|}/13]) \tag{4}$$

and (4) with the condition (3) is impossible for sufficiently large $|d|$ by a theorem of Gelfond [2, p. 34]. Unfortunately, Gelfond's theorem is noneffective in that it is unable to say what "sufficiently large" means and so Gelfond and Linnik have merely another version of Heilbronn's result that there are only finitely many complex quadratic fields of class-number one. Baker [1] makes Gelfond's theorem effective and in this way he reduced the class-number one problem to checking a finite number of discriminants.

But strangely enough, while Gelfond's general theorem is noneffective for linear forms in three or more logarithms, Gelfond was able to prove by other means an effective theorem for linear forms in two logarithms [2, p. 174]. Further, by using $k=12$ and 24 in (2) we may eliminate one logarithm:

$$h(24d)\log(5+2\sqrt{6})-2h(12d)\log(2+\sqrt{3}) = O(\exp[-\pi\sqrt{|d|}/24]). \tag{5}$$

Thus Gelfond and Linnik's idea coupled with Gelfond's effective theorem on linear forms in two logarithms would have settled the class-number one problem in 1949 had only the expansion in (2) been available with characters to nonprime moduli.

References

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