ON THE COMPACTNESS OF THE STRUCTURE
SPACE OF A RING

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1. Introduction. N. Jacobson [2, p. 204] has shown that a topology can be defined on the set $S(A)$ of primitive ideals of any nonradical ring $A$. With this topology, $S(A)$ is called the structure space of $A$. The topology is given by defining closure: If $T = \{p\}$ is a set of primitive ideals then the closure of $T$, $\text{Cl } T$ is the set of primitive ideals which contain

$$DT = \bigcap\{P | P \subseteq T\}.$$

It is well known that if $A$ has an identity element, then $S(A)$ is compact [2, p. 208]. Moreover, M. Schreiber [3] has observed that if every two-sided ideal of $A$ is finitely generated, then $S(A)$ is again compact. Further, R. L. Blair and L. C. Eggen [1] have obtained a result for a class of rings consisting of those $A$ such that

(C) no nonzero homomorphic image of $A$ is a radical ring

stating that the structure space of such a ring is compact if and only if $A$ is generated, as an ideal, by a finite number of elements.

The author found that for a certain class of rings, the modularity of the radical is both necessary and sufficient for the compactness of $S(A)$, and also that $S(A)$ is locally compact. For each $a \in A$, write $(a)$ for the principal two-sided ideal generated by $a$, and let

$$U_a = \{P | P \supseteq (a)\}.$$

Then $\{U_a\}$, $a \in A$, is an open basis of the topology [3]. The author is interested in a ring $A$ such that

(C') for every $a \in A$, $DU_a$ is modular.

A two-sided ideal $P$ is modular in the sense that there exists a two-sided identity modulo $P$.

2. Main results. Let $A$ be a ring with the property that $U_a$ is modular for every $a \in A$ (in this section).

Theorem 1. The structure space of $A$ is compact if and only if the radical $R$ of $A$ is modular.

Proof. Suppose $S(A)$ is compact. Since $\{U_x\}$, $x \in A$, is an open

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cover, there exists a subcover \( \{ U_a \} \), \( a \in E \), where \( E \) is a finite subset of \( A \). Then

\[
DU \cup \bigcup_{a \in E} U_a = DS(A) = R
\]

since \( \bigcup_{a \in E} U_a = S(A) \). But

\[
DU \cup \bigcup_{x \in A} \bigcap_{a \in E} DU_a.
\]

Hence

\[
R \supset \bigcap_{a \in E} DU_a.
\]

By hypothesis, each \( DU_a, a \in E \), is modular. Then it follows that the radical \( R \) of \( A \) is modular since an intersection of a finite number of modular two-sided ideals is modular.

Conversely, suppose \( R \) is modular with an identity \( e \) modulo \( R \). Then \( A/R \) is clearly a ring with an identity. Since \( A/R \) has an identity, the structure space \( S(A/R) \) is compact. Hence it follows that \( S(A) \) is compact because \( S(A) \) is homeomorphic to \( S(A/R) \) by the corollary in [2, p. 205].

**Theorem 2.** The structure space of \( A \) is locally compact.

**Proof.** Let \( P \) be a point of \( S(A) \) and take \( U_a \) as an open neighborhood of \( P \). Since \( DU_a \) is modular, \( S(A/DU_a) \) is compact. Therefore its homeomorphic image \( \text{Cl } U_a \) is compact and thus \( S(A) \) is locally compact.

3. **Examples.** A biregular ring \( A \), in the sense that if \( a \in A \) then there exists a central idempotent element \( e \) such that \( (a) = (e) \), satisfies the condition \((C')\), for it is easily seen that the element \( e \) is an identity modulo every primitive ideal that does not contain \((a)\). But the ring of integers satisfies the condition \((C')\) while it fails to be a biregular ring.

It is further investigated that the condition \((C')\) does not imply the condition \((C)\), and vice versa.

**Example 1.** Let \( B \) be a simple ring with an identity element and let \( R \) be a nonzero radical ring, and let \( A \) be the direct sum \( B \oplus R \). Then \( A \) has exactly one primitive ideal, namely, \( R \). Thus, for \( a \in A \), either

\[
DU_a = A \quad (\text{if } a \in R) \quad \text{or} \quad DU_a = R \quad (\text{if } a \notin R).
\]

In either case, \( DU_a \) is modular, so that \( A \) satisfies \((C')\). However, \( A/B \cong R \) is a radical ring, so \( A \) does not satisfy \((C)\).

**Example 2.** Let \( A \) be a simple ring without an identity element
and not a radical ring. Clearly $A$ satisfies (C). The only primitive ideal is the zero ideal $(0)$. If $a \in A$ with $a \neq 0$, then $DU_a = (0)$. Since $A$ has no identity element, $DU_a$ is not modular, so $A$ does not satisfy (C').

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**References**


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