ON THE COMPACTNESS OF THE STRUCTURE SPACE OF A RING

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1. **Introduction.** N. Jacobson [2, p. 204] has shown that a topology can be defined on the set $S(A)$ of primitive ideals of any nonradical ring A. With this topology, $S(A)$ is called the structure space of A. The topology is given by defining closure: If $T=\{P\}$ is a set of primitive ideals then the closure of T, Cl T is the set of primitive ideals which contain

$$DT = \bigcap\{P \mid P \in T\}.$$

It is well known that if A has an identity element, then $S(A)$ is compact [2, p. 208]. Moreover, M. Schreiber [3] has observed that if every two-sided ideal of A is finitely generated, then $S(A)$ is again compact. Further, R. L. Blair and L. C. Eggan [1] have obtained a result for a class of rings consisting of those A such that

(C) no nonzero homomorphic image of A is a radical ring

stating that the structure space of such a ring is compact if and only if A is generated, as an ideal, by a finite number of elements.

The author found that for a certain class of rings, the modularity of the radical is both necessary and sufficient for the compactness of $S(A)$, and also that $S(A)$ is locally compact. For each $a \in A$, write (a) for the principal two-sided ideal generated by a, and let

$$U_a = \{P \mid P \not\supseteq (a)\}.$$

Then $\{U_a\}$, $a \in A$, is an open basis of the topology [3]. The author is interested in a ring A such that

(C′) for every $a \in A$, DU_a is modular.

A two-sided ideal P is modular in the sense that there exists a two-sided identity modulo P.

2. **Main results.** Let A be a ring with the property that U_a is modular for every $a \in A$ (in this section).

THEOREM 1. *The structure space of A is compact if and only if the radical R of A is modular.*

PROOF. Suppose $S(A)$ is compact. Since $\{U_x\}$, $x \in A$, is an open

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cover, there exists a subcover $\{U_a\}$, $a \in E$, where E is a finite subset of A. Then

$$D\bigcup_{a\in E} U_a = DS(A) = R$$

since $\bigcup_{a\in E} U_a = S(A)$. But

$$D\bigcup_{x\in A} U_a \supseteq \bigcap_{a\in E} DU_a.$$

Hence

$$R \supseteq \bigcap_{a\in E} DU.$$

By hypothesis, each DU_a, $a \in E$, is modular. Then it follows that the radical R of A is modular since an intersection of a finite number of modular two-sided ideals is modular.

Conversely, suppose R is modular with an identity e modulo R. Then A/R is clearly a ring with an identity. Since A/R has an identity, the structure space $S(A/R)$ is compact. Hence it follows that $S(A)$ is compact because $S(A)$ is homeomorphic to $S(A/R)$ by the corollary in [2, p. 205].

THEOREM 2. *The structure space of A is locally compact.*

PROOF. Let P be a point of $S(A)$ and take U_a as an open neighborhood of P. Since DU_a is modular, $S(A/DU_a)$ is compact. Therefore its homeomorphic image Cl U_a is compact and thus $S(A)$ is locally compact.

3. **Examples.** A biregular ring A, in the sense that if $a \in A$ then there exists a central idempotent element e such that $(a) = (e)$, satisfies the condition (C′), for it is easily seen that the element e is an identity modulo every primitive ideal that does not contain (a). But the ring of integers satisfies the condition (C′) while it fails to be a biregular ring.

It is further investigated that the condition (C′) does not imply the condition (C), and vice versa.

EXAMPLE 1. Let B be a simple ring with an identity element and let R be a nonzero radical ring, and let A be the direct sum $B \oplus R$. Then A has exactly one primitive ideal, namely, R. Thus, for $a \in A$, either

$$DU_a = A \qquad (\text{if } a \in R) \quad \text{or} \quad DU_a = R \qquad (\text{if } a \notin R).$$

In either case, DU_a is modular, so that A satisfies (C′). However, $A/B \cong R$ is a radical ring, so A does not satisfy (C).

EXAMPLE 2. Let A be a simple ring without an identity element and not a radical ring. Clearly A satisfies (C). The only primitive ideal is the zero ideal (0). If $a \in A$ with $a \neq 0$, then $\boldsymbol{D}U_a = (0)$. Since A has no identity element, $\boldsymbol{D}U_a$ is not modular, so A does not satisfy (C′).

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