INDUCED REPRESENTATIONS OF LIE ALGEBRAS. II

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1. **Introduction.** Let $\mathfrak{G}$ be a Lie algebra over a field K. A decomposition of $\mathfrak{G}$ is a triple $(\mathfrak{n}_1, \mathfrak{h}, \mathfrak{n}_2)$ of subalgebras of $\mathfrak{G}$ such that $\mathfrak{G}=\mathfrak{n}_1\oplus\mathfrak{h}\oplus\mathfrak{n}_2$ a vector space direct sum and such that $[\mathfrak{h}, \mathfrak{n}_i]\subset\mathfrak{n}_i$ for $i=1, 2$. In [5] we showed how one could "induce" $\mathfrak{G}$-modules from $\mathfrak{h}$-modules in a natural manner. In this paper we prove a useable necessary and sufficient condition that a $\mathfrak{G}$-module must satisfy in order that it be "induced" from an $\mathfrak{h}$-module. We apply this method of induction to obtain all well-known simple modules (not necessarily finite dimensional) for semisimple Lie algebras over algebraically closed fields of characteristic 0.

2. **Preliminary results.** Let $\mathfrak{G}$ be a Lie algebra over a field K with decomposition $(\mathfrak{n}_1, \mathfrak{h}, \mathfrak{n}_2)$. Set $\mathfrak{k}=\mathfrak{n}_1+\mathfrak{h}$. Let $U(\mathfrak{G})$ be the universal enveloping algebra of $\mathfrak{G}$ and let $U(\mathfrak{k})$, $U(\mathfrak{n}_1)$, $U(\mathfrak{h})$ and $U(\mathfrak{n}_2)$ be the universal enveloping algebras of $\mathfrak{k}$, $\mathfrak{n}_1$, $\mathfrak{h}$, $\mathfrak{n}_2$ canonically embedded in $U(\mathfrak{G})$. We look upon $U(\mathfrak{G})$ as a left $U(\mathfrak{k})$-module and a right $U(\mathfrak{G})$-module under, respectively, left and right multiplication. Let W be an $\mathfrak{h}$-module. We denote by $\hat{W}$ the $\mathfrak{k}$-module with space W and with $\mathfrak{n}_1\cdot W=0$. We set $T(W)=\mathrm{Hom}_{U(\mathfrak{k})}(U(\mathfrak{G}), \hat{W})$ with the natural left $\mathfrak{G}$-module structure

$$(x\cdot f)(g) = f(gx) \qquad \text{for } x\in\mathfrak{G},\ \ f\in T(W),\ \ g\in U(\mathfrak{G}).$$

By the Poincaré-Birkhoff-Witt (abbreviated P-B-W) theorem $U(\mathfrak{G}) = U(\mathfrak{k})\mathfrak{n}_2\cdot U(\mathfrak{n}_2)\oplus U(\mathfrak{k})$ a right $\mathfrak{h}$-module, left $\mathfrak{k}$-module direct sum. Let γ: $U(\mathfrak{G})\rightarrow U(\mathfrak{k})$ be the corresponding $\mathfrak{h}$-module projection. Let w: $W\rightarrow T(W)$ be defined by $w(v)(g)=\gamma(g)\cdot v$ for $g\in U(\mathfrak{G})$, $v\in W$. It is easy to see that $w(W)=T(W)\circ\gamma$ (see [5]) and is thus isomorphic with W as an $\mathfrak{h}$-module. We set $W^*=U(\mathfrak{G})\cdot w(W)$. W^* is a $\mathfrak{G}$-submodule of $T(W)$ and W^* is the "induced" module of §1. Before proceeding, we introduce one more bit of notation. If V is a left $\mathfrak{G}$-module and if M is a subalgebra of $\mathfrak{G}$ then set $V^M=\{v\in V \mid M\cdot v=0\}$.

LEMMA 2.1. *Let W be an $\mathfrak{h}$-module.*

(1) $(W^*)^{\mathfrak{n}_2}\neq(0)$ if $W^*\neq(0)$.

(2) $U(\mathfrak{G})\cdot(W^*)^{\mathfrak{n}_2}=W^*$.

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(3) $(W^*)^{\mathfrak{n}_2} \cap \mathfrak{n}_1 \cdot W^* = (0)$.

(4) $\mathfrak{n}_1 \cdot W^*$ *contains no nonzero* $\mathfrak{G}$*-modules.*

PROOF. $w(\hat{W}) = (W^*)^{\mathfrak{n}_2}$ and thus (1) is true. Statement (2) is just the definition of W^*. To prove statement (3) we notice that if $f \in (W^*)^{\mathfrak{n}_2}$ then $f \neq 0$ if and only if $f(1) \neq 0$. If $n_1 \cdot f \in (W^*)^{\mathfrak{n}_2}$ for some $f \in W^*$ and $n_1 \in \mathfrak{n}_1$ then $(n_1 \cdot f)(1) = f(n_1) = n_1 \cdot f(1) = 0$. Thus $n_1 \cdot f = 0$. We have thus proven (3). If $f \in \mathfrak{n}_1 \cdot W^*$ and if $U(\mathfrak{G}) \cdot f \subset \mathfrak{n}_1 \cdot W^*$ then $(g \cdot f) \circ \gamma = 0$ for all $g \in U(\mathfrak{G})$. In fact, if $(g \cdot f) \circ \gamma \neq 0$ for some $g \in U(\mathfrak{G})$ then $g \cdot f \circ \gamma \in (W^*)^{\mathfrak{n}_2}$ (by the remarks preceding the lemma) and $g \cdot f \in \mathfrak{n}_1 \cdot W^*$ implies that $g \cdot (f \circ \gamma) \in \mathfrak{n}_1 \cdot W^* \cap (W^*)^{\mathfrak{n}_2} = (0)$, which is a contradiction. We thus have $0 = (g \cdot f)(\gamma(1)) = (g \cdot f)(1) = f(g)$ for all $g \in U(\mathfrak{G})$. Hence $f = 0$. The lemma is thus proved.

Lemma 1 implies that as an $\mathfrak{h}$-module, $W^* = (W^*)^{\mathfrak{n}_2} \oplus \mathfrak{n}_1 \cdot W^*$. Let $P: W^* \to (W^*)^{\mathfrak{n}_2}$ and $Q: W^* \to \mathfrak{n}_1 \cdot W^*$ be the corresponding $\mathfrak{h}$-module projections.

LEMMA 2.2. *Let V be a $\mathfrak{G}$-module and let W be an $\mathfrak{h}$-module. Then the map* $\mathrm{Hom}_{U(\mathfrak{G})}(V, W^*) \to \mathrm{Hom}_{U(\mathfrak{h})}(V, W)$ *given by* $f \to P \circ f$ *is* 1-1. (*Here $W^{*\mathfrak{n}_2}$ is identified with W as an $\mathfrak{h}$-module.*)

PROOF. If $P \circ f = 0$ then $f(V) \subset \mathfrak{n}_1 \cdot W^*$. Lemma 1 (4) tell us that $f(V) = (0)$. Thus $f = 0$.

We note that Lemma 2 is formally half of the Frobenius reciprocity theorem. We conclude this section with a useful sufficient condition for simplicity of W^*.

PROPOSITION 2.1. *Let W be a simple $\mathfrak{h}$-module. Suppose that $\mathfrak{n}_1 \cdot W^*$ contains no $\mathfrak{h}$-modules isomorphic with W. Then W^* is simple.*

PROOF. Let M be a nonzero $\mathfrak{G}$-submodule of W^*. Let P and Q be as above. By Lemma 2.1 (4), $P(M) \neq (0)$. Thus $P(M) = W^{*\mathfrak{n}_2}$ since $W^{*\mathfrak{n}_2}$ is simple. If $M \neq W^*$ then there is a $v \in M$ such that $P(v) \neq 0$ and $Q(v) \neq 0$. In fact, if there is a $v \in M$ such that $Q(v) = 0$, $P(v) \neq 0$ then $v \in W^{*\mathfrak{n}_2}$ and by definition of W^* and simplicity of W, $U(\mathfrak{G}) \cdot v = W^*$. Let $v \in M$ such that $P(v) \neq 0$ and $Q(v) \neq 0$. If $h \cdot P(v) \neq 0$ and $h \cdot Q(v) = 0$ for some $h \in U(\mathfrak{h})$ then by the above argument $M = W^*$. Thus there is an element v in M such that $P(v) \neq 0$ and such that if $h \cdot P(v) \neq 0$ then $h \cdot Q(v) \neq 0$. Now let $\tilde{W} = U(\mathfrak{h}) \cdot Q(v) \subset \mathfrak{n}_1 \cdot W^*$. We define a map $\xi: \tilde{W} \to W^{*\mathfrak{n}_2}$ by setting $\xi(h \cdot Q(v)) = h \cdot P(v)$ for each $h \in U(\mathfrak{h})$. We show that ξ is well defined. If $h \cdot Q(v) = h' \cdot Q(v)$ then $(h - h')Q(v) = 0$. Thus $(h - h')P(v) = 0$ and thus $\xi(h \cdot Q(v)) = \xi(h' \cdot Q(v))$. Thus ξ is well defined and injective. ξ is now clearly an $\mathfrak{h}$-module isomorphism. This contradiction implies that $M = W^*$.

3. **An imprimitivity theorem.** In this section we prove the converse of Lemma 2.1. We maintain the notation of §2.

THEOREM 3.1. *If V is a $\mathfrak{G}$-module such that*
(1) $V^{\mathfrak{n}_2} \neq (0)$ *if* $V \neq (0)$,
(2) $U(\mathfrak{G}) \cdot V^{\mathfrak{n}_2} = V$,
(3) $\mathfrak{n}_1 \cdot V \cap V^{\mathfrak{n}_2} = (0)$,
(4) $\mathfrak{n}_1 \cdot V$ *contains no nonzero $\mathfrak{G}$-submodules of V then V is $\mathfrak{G}$-isomorphic with* $(V^{\mathfrak{n}_2})^*$.

PROOF. By the P-B-W theorem $U(\mathfrak{G}) = U(\mathfrak{n}_2 + \mathfrak{h}) + \mathfrak{n}_1 \cdot U(\mathfrak{n}_1) U(\mathfrak{n}_2 + \mathfrak{h})$ a left $\mathfrak{h}$-module direct sum. Let γ_1: $U(\mathfrak{G}) \to U(\mathfrak{n}_2 + \mathfrak{h})$ and γ_2: $U(\mathfrak{G}) \to \mathfrak{n}_1 \cdot U(\mathfrak{n}_1) U(\mathfrak{n}_2 + \mathfrak{h})$ be the corresponding $\mathfrak{h}$-module projections. Suppose that $v \in V$. Then $v = g \cdot \bar{v}$ for some $\bar{v} \in V^{\mathfrak{n}_2}$ by (2). Thus $v = \gamma_1(g) \cdot \bar{v} + \gamma_2(g) \cdot \bar{v}$ with $\gamma_1(g) \cdot \bar{v} \in V^{\mathfrak{n}_2}$ and $\gamma_2(g) \cdot \bar{v} \in \mathfrak{n}_1 \cdot V$. Now (3) implies that $V = V^{\mathfrak{n}_2} \oplus \mathfrak{n}_1 \cdot V$ an $\mathfrak{h}$-module direct sum. Let R: $V \to V^{\mathfrak{n}_2}$ be the corresponding $\mathfrak{h}$-module projection. We define δ: $V \to T(V^{\mathfrak{n}_2})$ by $\delta(v)(g) = R(g \cdot v)$. Then $\delta(v)(kg) = R(kg \cdot v) = 0$ if $k \in \mathfrak{n}_1 \cdot U(\mathfrak{G})$. $\delta(v)(kg) = R(kg \cdot v) = k \cdot R(gv)$ if $k \in \mathfrak{h}$. Thus $\delta(v) \in T(W)$. Now $\delta(g_0 \cdot v)(g) = R(gg_0 \cdot v) = \delta(v)(gg_0) = (g_0 \cdot \delta(v))(g)$ and thus δ: $V \to T(V)$ is a $\mathfrak{G}$-module homomorphism. If $\delta(v) = 0$ then $\delta(v)(g) = 0$ for all $g \in U(\mathfrak{G})$ and thus $R(g \cdot v) = 0$ for all $g \in U(\mathfrak{G})$. This says that $U(\mathfrak{G}) \cdot v \subset \mathfrak{n}_1 \cdot V$ and hence $v = 0$. Thus δ is injective. Suppose that $v \in V^{\mathfrak{n}_2}$ then $\delta(v)(g) = R(g \cdot v) = \gamma(g) \cdot v$ (here v is looked upon as an element of $V^{\mathfrak{n}_2})\hat{}$. Thus $\delta|_{V^{\mathfrak{n}_2}} = w$. This clearly implies that

$$\delta(V) = U(\mathfrak{G}) w(V^{\mathfrak{n}_2}) = (V^{\mathfrak{n}_2})^*. \qquad \text{Q.E.D.}$$

As a corollary to Theorem 3.1 we derive the main result of [5] without using the technique of "double dualization."

COROLLARY 3.1. *Suppose that V is a finite dimensional simple $\mathfrak{G}$-module and that $\mathfrak{n}_1$ and $\mathfrak{n}_2$ act nilpotently on V. Then V is isomorphic with $(V^{\mathfrak{n}_2})^*$ as a $\mathfrak{G}$-module.*

PROOF. We show that V satisfies conditions (1)–(4) of Theorem 3.1. If $V = (0)$ then $V = (0)^*$. If $V \neq (0)$ then by assumption $V^{\mathfrak{n}_2} \neq (0)$. Since V is simple, $U(\mathfrak{G}) \cdot V^{\mathfrak{n}_2} = V$. If $\mathfrak{n}_1 \cdot V \cap V^{\mathfrak{n}_2} \neq (0)$ then

$$V = U(\mathfrak{G}) \cdot (\mathfrak{n}_1 \cdot V \cap V^{\mathfrak{n}_2}) = U(\mathfrak{n}_1) \cdot (\mathfrak{n}_1 \cdot V \cap V^{\mathfrak{n}_2}) \subset U(\mathfrak{n}_1) \mathfrak{n}_1 \cdot V.$$

But $\mathfrak{n}_1$ acts nilpotently on V; thus the above inclusion implies the contradiction $V = (0)$. Thus $\mathfrak{n}_1 \cdot V \cap V^{\mathfrak{n}_2} = (0)$. Finally, since $\mathfrak{n}_1 \cdot V$ is a proper subspace of V, $\mathfrak{n}_1 \cdot V$ cannot contain any nonzero $\mathfrak{G}$-submodule of V.

4. The standard decomposition of a semisimple Lie algebra. In this section we assume that $\mathfrak{G}$ is semisimple and that K is of characteristic 0 and algebraically closed. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{G}$ and let Δ be the root system of $\mathfrak{G}$ with respect to $\mathfrak{h}$. That is, let Δ be the set of all linear forms α on $\mathfrak{h}$ such that the set $\mathfrak{G}_\alpha = \{x \in \mathfrak{G} \mid [h, x] = \alpha(h) \cdot x \text{ for all } h \in \mathfrak{h}\}$ is nonzero. Let π be a set of linearly independent elements of Δ such that every element of Δ can be written as an integral combination of the elements of π with the coefficients all having the same sign. Such systems always exist, see e.g. Jacobson [**3**]. Let $\succ$ be a linear order on Δ corresponding to π (i.e. $\alpha \succ 0$ if $\alpha = \sum n_\gamma \gamma$ sum over $\gamma \in \pi$ with $\sum n_\gamma > 0$). Let $\mathfrak{n}^+ = \sum_{\alpha \succ 0} \mathfrak{G}_\alpha$, $\mathfrak{n}^- = \sum_{\alpha \prec 0} \mathfrak{G}_\alpha$. Then $\mathfrak{G} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ is a decomposition of $\mathfrak{G}$.

PROPOSITION 4.1. *If W is a simple finite dimensional* ($=1$ *dimensional*) *$\mathfrak{h}$-module then W^* is a simple $\mathfrak{G}$-module.*

PROOF. $W^* = U(\mathfrak{G}) \cdot W^{*\mathfrak{n}^+} = U(\mathfrak{n}^-) \cdot W^{*\mathfrak{n}^+} = W^{*\mathfrak{n}^+} \oplus \mathfrak{n}^- \cdot U(\mathfrak{n}^-) \cdot W^{*\mathfrak{n}^+}$. Now consider $\mathfrak{n}^- \cdot U(\mathfrak{n}^-)$ as an $\mathfrak{h}$-module under the action $\mathrm{ad}(h)n = h \cdot n - n \cdot h$ for $n \in \mathfrak{n}^- \cdot U(\mathfrak{n}^-)$.

By the P-B-W theorem $\mathfrak{n}^- \cdot U(\mathfrak{n}^-)$ is completely reducible as an $\mathfrak{h}$-module and $\mathfrak{n}^- \cdot U(\mathfrak{n}^-) = \sum V_\gamma$ where $V_\gamma = \{n \in \mathfrak{n}^- \cdot U(\mathfrak{n}^-) \mid \mathrm{ad}(h) \cdot n = \gamma(h) \cdot n,\ h \in \mathfrak{h}\}$ and the above sum is an $\mathfrak{h}$-module direct sum where the γ's are taken to be all nonnegative integral combinations of positive roots. Furthermore $\dim_K V_\gamma = p(\gamma)$. $p(\gamma)$ is the number of ways that γ can be written as a sum $\gamma = \gamma_1 + \cdots + \gamma_r$ where $\gamma_i \in \Delta$, $0 \prec \gamma_1 \preceq \gamma_2 \preceq \gamma_3 \preceq \cdots \preceq \gamma_r$. Thus $\mathfrak{n}^- \cdot U(\mathfrak{n}^-) \otimes_K W^{*\mathfrak{n}^+}$ as a Lie algebra tensor product module (i.e. $h \cdot (n \otimes v) = \mathrm{ad}(h) \cdot n \otimes v + n \otimes h \cdot v$) is isomorphic with $\sum_\gamma p(\gamma) K_\gamma \otimes W$ where K_γ is the $\mathfrak{h}$-module K with action $h \cdot 1 = \gamma(h)$, $p(\gamma)$ is the corresponding multiplicity. $K_\gamma \otimes W$ is a simple $\mathfrak{h}$-module. Let $\phi: \mathfrak{n}^- \cdot U(\mathfrak{n}^-) \otimes W^{*\mathfrak{n}^+} \to \mathfrak{n}^- \cdot U(\mathfrak{n}^-) \cdot W^{*\mathfrak{n}^+}$ by $\phi(n \otimes v) = n \cdot v$ then ϕ is an $\mathfrak{h}$-module homomorphism. Thus $\mathfrak{n}^- \cdot W^* = \mathfrak{n}^- \cdot U(\mathfrak{n}^-) \cdot W^{*\mathfrak{n}^+}$ is $\mathfrak{h}$-isomorphic with $\sum_\gamma M_\gamma K_\gamma \otimes W$ where the M_γ correspond as above to multiplicities. Let $P_\gamma: \mathfrak{n}^- \cdot W^* \to M_\gamma K_\gamma \otimes W$ be the corresponding $\mathfrak{h}$-module projection. Suppose that $\tilde{W}$ is an $\mathfrak{h}$-submodule of $\mathfrak{n}^- \cdot W$ such that $\tilde{W}$ is $\mathfrak{h}$-isomorphic with W. Let γ be such that $P_\gamma(\tilde{W}) \neq 0$. Since $\tilde{W}$ is simple $P_\gamma|_{\tilde{W}}$ is an $\mathfrak{h}$-module injection. Thus W is $\mathfrak{h}$-isomorphic with $K_\gamma \otimes W$, for some $\gamma \neq 0$, which is impossible. Thus W^* is simple by Theorem 3.1.

If W is the $\mathfrak{h}$-module K_λ where $\lambda \in \mathfrak{h}^0$, the dual space of $\mathfrak{h}$, and K_λ is the $\mathfrak{h}$-module K with the action $h \cdot 1 = \lambda(h)$, then we set $W^* = V^\lambda$. We notice that $(V^\lambda)^{\mathfrak{n}^+} = \{v \in V^\lambda \mid h \cdot v = \lambda(h) v,\ h \in \mathfrak{h}\} = w(K_\lambda)$ and thus $\dim (V^\lambda)^{\mathfrak{n}^+} = 1$. Furthermore $V^\lambda = U(\mathfrak{G}) \cdot (V^\lambda)^{\mathfrak{n}^+} = U(\mathfrak{n}^-) \cdot (V^\lambda)^{\mathfrak{n}^+}$. Thus by the techniques of the proof of Proposition 4.1 we find that V^λ

$=\sum_{\mu\in\mathfrak{h}^0} V^\lambda_\mu$ where $V^\lambda_\mu=\{v\in V^\lambda \mid h\cdot v=\mu(h)v\}$. And $V^\lambda_\mu\neq 0$ only if $\lambda-\mu$ is a sum of positive roots. Thus V^λ is the $\mathfrak{n}^+$ extreme $\mathfrak{h}$-module with highest weight λ as in Jacobson [3] or Sophus Lie [4].

5. A multiplication on $U(\mathfrak{G})^0$. Let $\mathfrak{G}$ be a Lie algebra over a field K. Let $U(\mathfrak{G})$ be its universal enveloping algebra and let $U(\mathfrak{G})^0$ be the dual space of $U(\mathfrak{G})(=\operatorname{Hom}_K(U(\mathfrak{G}), K))$. Let ϵ: $U(\mathfrak{G})^0\otimes U(\mathfrak{G})^0 \to K$ be given by $\epsilon(f\otimes g)=f(1)g(1)$ and extended by linearity. $U(\mathfrak{G})^0$ is a left $\mathfrak{G}$-module under the action $(x\cdot f)(g)=f(gx)$ where $f\in U(\mathfrak{G})^0$, $x\in\mathfrak{G}$, $g\in U(\mathfrak{G})$. We look upon $U(\mathfrak{G})^0\otimes U(\mathfrak{G})^0$ as the tensor product $\mathfrak{G}$-module (hence $U(\mathfrak{G})$-module). We can now define a multiplication on $U(\mathfrak{G})^0$: let $f, f'\in U(\mathfrak{G})^0$ then $f\cdot f'(g)=\epsilon(g\cdot(f\otimes f'))$. It is not hard to check that $U(\mathfrak{G})^0$ is an associative, commutative ring under this multiplication. $U(\mathfrak{G})^0$ contains a unit. In fact, $U(\mathfrak{G})=K\cdot 1\oplus\mathfrak{G}\cdot U(\mathfrak{G})$ let f_0: $U(\mathfrak{G})\to K$ be the corresponding projection. Then $f_0\cdot f=f$ for all $f\in U(\mathfrak{G})^0$. If K is of characteristic 0 then $U(\mathfrak{G})^0$ is an integral domain. (If K is of characteristic $p\neq 0$ then $U(\mathfrak{G})^0$ has no elements f such that $f^2=f$.) For details see Cartier [1] or Hochschild [2]. We also note that the map $U(\mathfrak{G})^0\otimes U(\mathfrak{G})^0\to U(\mathfrak{G})^0$ given by $f\otimes f'\to f\cdot f'$ is a $\mathfrak{G}$-module homomorphism. Thus $\mathfrak{G}$ acts on $U(\mathfrak{G})^0$ as derivations.

We now return to the case $\mathfrak{G}$ semisimple and K characteristic 0 and algebraically closed, and we maintain the notation of §4. If V is a $\mathfrak{G}$-module and $\mu\in\mathfrak{h}^0$ we set (as before) $V_\mu=\{v\in V\mid h\cdot v=\mu(h)\cdot v\}$. We also note that if $\lambda\in\mathfrak{h}^0$ then (setting) $T(K_\lambda)=T(\lambda)\subset U(\mathfrak{G})^0$. Set ${}^{\mathfrak{n}^-}U(\mathfrak{G})^0=\{f\in U(\mathfrak{G})^0 \mid f(ng)=0 \text{ for all } n\in\mathfrak{n}^-, g\in U(\mathfrak{G})\}$.

LEMMA 5.1. (a) ${}^{\mathfrak{n}^-}U(\mathfrak{G})^0$ *is a subring of* $U(\mathfrak{G})^0$.

(b) *If* $\lambda, \lambda', \mu, \mu'\in\mathfrak{h}^0$ *then* $T(\lambda)_\mu\cdot T(\lambda')_{\mu'}\subset T(\lambda+\lambda')_{\mu+\mu'}$.

PROOF. (a) Let $n\in\mathfrak{n}^-$, $g\in U(\mathfrak{G})$, $f, f'\in{}^{\mathfrak{n}^-}U(\mathfrak{G})^0$. Then $f\cdot f'(ng)$ $=\epsilon(ng(f\otimes f'))$. Let $g\cdot(f\otimes f')=\sum_{ij} f_i\otimes f'_j, f'_j, f_i\in{}^{\mathfrak{n}^-}U(\mathfrak{G})^0$ (if $f\in{}^{\mathfrak{n}^-}U(\mathfrak{G})^0$ then $g\cdot f\in{}^{\mathfrak{n}^-}U(\mathfrak{G})^0$). $\epsilon(ng(f\otimes f'))=\epsilon(n\cdot\sum f_i\otimes f'_j)$ $=\epsilon(\sum(n\cdot f_i\otimes f'_j+f_i\otimes n\cdot f'_j))=\sum(f_i(n)\cdot f'_j(1)+f_i(1)f'_j(n))=0$. Thus $f\cdot f'\in{}^{\mathfrak{n}^-}U(\mathfrak{G})^0$.

(b) is proved similarly.

Let $F(\mathfrak{G})=\{f\in{}^{\mathfrak{n}^-}U(\mathfrak{G})^0 \mid \dim_K U(\mathfrak{G})\cdot f<\infty\}$. Clearly $F(\mathfrak{G})$ is a subalgebra of ${}^{\mathfrak{n}^-}U(\mathfrak{G})^0$ and $f_0\in F(\mathfrak{G})$.

THEOREM 5.1. $F(\mathfrak{G})$ *contains every finite dimensional simple* $\mathfrak{G}$-*module exactly once. Furthermore*, $F(\mathfrak{G})=\sum_\lambda V^\lambda$ *a* $\mathfrak{G}$-*module direct sum over all* λ *dominant integral* (*that is if* $\alpha\in\pi$ *then* $2\langle\lambda, \alpha\rangle/\langle\alpha, a\rangle$ *is a nonnegative integer*). *Furthermore* $V^\lambda\cdot V^{\lambda'}=V^{\lambda+\lambda'}$ *if* λ *and* λ' *are dominant integral.*

PROOF. Let V be a finite dimensional simple submodule of $F(\mathfrak{G})$. Then $\dim_K V^{\mathfrak{n}^+}=1$ and there is a dominant integral $\lambda\in\mathfrak{h}^0$ such that if $v\in V^{\mathfrak{n}^+}$ then $h\cdot v=\lambda(h)\cdot v$. If $g\in U(\mathfrak{G})$, $h\in\mathfrak{h}$, $f\in V^{\mathfrak{n}^+}$ then $g=n^-h'n^+$, $n^-\in U(\mathfrak{n}^-)$, $h'\in U(\mathfrak{h})$, $n^+\in U(\mathfrak{n}^+)$ and $f(hg)=f(hn^-h'n^+)=f(n^-hh'n^+)=0$ if $n^-\neq 1$ and $n^+\neq 1$. Thus if n^- or $n^+\neq 1$ then $f(hg)=0=\lambda(h)f(g)$. If $n^-=n^+=1$ then $f(hh')=f(h'h)=(h\cdot f)(h')=\lambda(h)f(h')$. Thus $f\in T(\lambda)^{\mathfrak{n}^+}=(V^\lambda)^{\mathfrak{n}^+}$. And thus $V=V^\lambda$.

If $f\in F(\mathfrak{G})$ then $\dim_K U(\mathfrak{G})\cdot f<\infty$. Thus $U(\mathfrak{G})\cdot f=\sum V_i$, V_i finite dimensional simple. Thus $f\in\sum V^\lambda$. Hence $F(\mathfrak{G})=\sum V^\lambda$, λ dominant integral.

By Lemma 5.1 we have $V^\lambda\cdot V^{\lambda'}\subset T(\lambda+\lambda')$. By the above the only finite dimensional simple submodule of $T(\lambda+\lambda')$ is $V^{\lambda+\lambda'}$. Thus if λ, λ' are dominant integral then $V^\lambda\cdot V^{\lambda'}=V^{\lambda+\lambda'}$.

Let $\pi=\{\alpha_1,\cdots,\alpha_l\}$ and let $\lambda_1,\cdots,\lambda_l$ be defined by $2\langle\alpha_i,\lambda_j\rangle/\langle\alpha_i,\alpha_i\rangle=\delta_{ij}=0$ if $i\neq j$, $=1$ if $i=j$. Then by Theorem 5.1 we know that $V^{\lambda_1}+V^{\lambda_2}+\cdots+V^{\lambda_l}$ generates $F(\mathfrak{G})$. Let $V=V^{\lambda_1}+V^{\lambda_2}+\cdots+V^{\lambda_l}$. Consider the symmetric algebra on V, $S(V)$, l-graded in the natural manner. That is $S^n(V)$ where $n=(n_1,\cdots,n_l)$, $n_i\geqq 0$, n_i an integer is the space spanned by all products of n_1 elements of V^{λ_1}, n_2 elements of V^{λ_2}, etc. $S(V)$ inherits a $\mathfrak{G}$-module structure from V. And there is a natural algebra and $\mathfrak{G}$-homomorphism Φ: $S(V)\rightarrow F(\mathfrak{G})$ such that $S^n(V)\rightarrow V^{n_1\lambda_1+\cdots+n_l\lambda_l}$. Since $F(\mathfrak{G})$ is an integral domain $\operatorname{Ker}\Phi$ is a prime ideal in $S(V)$. If $I(V)=\{f\in S(V)\mid x\cdot f=0$ for all $x\in\mathfrak{G}\}$ then $I^+(V)=I(V)\cap_{n\neq 0}S^n(V)$ is a subalgebra of $\operatorname{Ker}\Phi$. We have not as yet found the relationship between $\operatorname{Ker}\Phi$ and $I^+(V)$.

We conclude this section with a simple example of $\operatorname{Ker}\Phi$. Let $\mathfrak{G}=A_2\cdot\pi=\{\alpha_1,\alpha_2\}$. Then V^{λ_1} is just the 3 dimensional representation of A_2 as $\mathrm{sl}(3,K)$ and V^{λ_2} is just the dual module of V^{λ_1}. If $X_1\in V^{\lambda_1}_{\lambda_1}$, $X_2\in V^{\lambda_1}_{\lambda_1-\alpha_1}$, $X_3\in V^{\lambda_1}_{\lambda_1-\alpha_1-\alpha_2}$. Identifying V^{λ_2} with the dual module of V^{λ_1} we take the dual basis Y_1, Y_2, Y_3, of X_1, X_2, X_3. ($Y_1\in V^{\lambda_2}_{-\lambda_1}$, $Y_2\in V^{\lambda_2}_{\lambda_2-\alpha_2}$, $Y_3\in V^{\lambda_2}_{\lambda_2}$) and $\operatorname{Ker}\Phi$ is generated by $X_1Y_1+X_2Y_2+X_3Y_3$.

BIBLIOGRAPHY

1. P. Cartier, *Dualité de Tannaka des groupes et des algèbres de Lie*, C. R. Acad. Sci. Paris **242** (1956), 322–325.

2. G. Hochschild, *Algebraic Lie algebras and representative functions*, Illinois J. Math. **3** (1959), 499–523.

3. N. Jacobson, *Lie algebras*, Interscience, New York, 1962.

4. Séminaire Sophus Lie, Mimeographed notes, Ann. Sci. École Norm. Sup., Paris, 1954–1955.

5. N. Wallach, *Induced representations of Lie algebras and a theorem of Borel-Weil*, Trans. Amer. Math. Soc. **136** (1969), 181–187.

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