DECOMPOSITION THEOREMS FOR VECTOR MEASURES

JAMES K. BROOKS

1. **Introduction.** In this note we consider two theorems for vector measures, each of which is a generalization of a well-known theorem for scalar measures. Theorem 1 extends the Yosida-Hewitt theorem [**7**], [**4**, p. 163], which states that a bounded real valued finitely additive (f.a.) measure defined on a Boolean algebra can be decomposed uniquely into a countably additive (c.a.) part and a purely finitely additive (p.f.a.) part. The second theorem, due to Rickart [**6**, Theorem 4.5], is a Lebesgue decomposition theorem for c.a. vector measures with respect to an outer measure. Several authors have given alternate proofs of a restricted form of this theorem (e.g. see [**3**, p. 189] where the outer measure is replaced by a measure and the total variation of the vector measure is assumed to be finite—a condition not generally satisfied). In 3 we give a short and elementary proof of Rickart's theorem, which represents a considerable simplification of the existing proofs of this result.

Let $\boldsymbol{X}$ be a Banach space over the reals $\boldsymbol{R}$ with first and second conjugate spaces $\boldsymbol{X}^*$ and $\boldsymbol{X}^{**}$; we regard $\boldsymbol{X}$ as a subset of $\boldsymbol{X}^{**}$. Σ_0 and Σ respectively denote a Boolean algebra and a σ-algebra of subsets of a set S. $\|\mu\|$ denotes the semivariation [**3**, p. 51] of μ. μ is *bounded* if $\|\mu\|(S)<\infty$.

2. THEOREM 1. *Let $\mu:\Sigma_0\to\boldsymbol{X}$ be bounded and finitely additive. Then μ can be written uniquely in the form $\mu=\mu_1+\mu_2$, where the $\mu_i:\Sigma_0\to\boldsymbol{X}^{**}$ are finitely additive and for each $f\in\boldsymbol{X}^*$:* (1)*$\mu_1(\cdot)f:\Sigma_0\to\boldsymbol{R}$ is countably additive and* (2) *$\mu_2(\cdot)f:\Sigma_0\to\boldsymbol{R}$ is purely finitely additive.*

PROOF. For $f\in\boldsymbol{X}^*$ the set function $f\mu$ defined by $(f\mu)(E)=f(\mu(E))$, $E\in\Sigma_0$ is f.a. Moreover, since $|f\mu(E)|\leqq\|f\|\,\|\mu(E)\|\leqq\|f\|\,\|\mu\|(S)$, $f\mu$ is bounded. By the Yosida-Hewitt theorem $f\mu=\mu_{f,1}+\mu_{f,2}$, where $\mu_{f,1}$ is c.a. and $\mu_{f,2}$ is p.f.a. For $E\in\Sigma_0$ define $F_{i,E}:\boldsymbol{X}^*\to\boldsymbol{R}$, $i=1, 2$, as follows: $F_{i,E}(f)=\mu_{f,i}(E)$, $f\in\boldsymbol{X}^*$. It follows from the uniqueness of the decomposition of the scalar measures and Theorem 1.17 in [**7**] that $\mu_{\alpha f,i}=\alpha\mu_{f,i}$ and $\mu_{f+g,i}=\mu_{f,i}+\mu_{g,i}$, for f, $g\in\boldsymbol{X}^*$, $\alpha\in\boldsymbol{R}$; hence $F_{i,E}$ is linear on $\boldsymbol{X}^*$. To show that $F_{i,E}\in\boldsymbol{X}^{**}$, let $f\in\boldsymbol{X}^*$ and let $f\mu=(f\mu)^+-(f\mu)^-$, $|f\mu|=(f\mu)^++(f\mu)^-$ be the Jordan decomposition of $f\mu$. Again using the uniqueness of the decomposition of $f\mu$, we have $u_{f,i}=(f\mu)_i^+-(f\mu)_i^-$ where the subscripts refer to the c.a. and p.f.a. parts. $|F_{i,E}(f)|$

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$= |\mu_{f,i}(E)| \leqq (f\mu)_i^+(E) + (f\mu)_i^-(E) \leqq (f\mu)^+(E) + (f\mu)^-(E) = |f\mu|(E) \leqq 2 \sup_{A \subseteq E} |f\mu(A)| \leqq 2 \sup_{A \subseteq E} \|f\| \|\mu(A)\| \leqq 2\|f\| \|\mu\|(E)$. Thus $\|F_{i,E}\| \leqq 2\|\mu\|(E)$ and $F_{i,E} \in X^{**}$. Define $\mu_i(E) = F_{i,E}$, $E \in \Sigma_0$. It is clear from the construction that μ_1 and μ_2 have the required properties.

COROLLARY. *Assume X is reflexive and $\mu: \Sigma \to X$ is bounded and finitely additive. Then μ can be written uniquely in the form $\mu = \mu_1 + \mu_2$, $\mu_i: \Sigma \to X$, where (1′) μ_1 is countably additive and (2′) $f\mu_2$ is purely finitely additive for each $f \in X^*$.*

PROOF. Let μ_i be the components of μ given in the above theorem. Since $f\mu_1(\cdot) = \mu_1(\cdot)f$ is c.a. for $f \in X^*$, μ_1 is weakly countably additive in addition to being f.a. By the Pettis theorem [4, p. 318] μ_1 is c.a., and the result follows.

REMARK. For a reflexive space X, the existence of a large number of bounded f.a. X-valued measures on Σ, hence measures satisfying (2′), can be deduced as follows. Let $Y = M_{X^*}(\Sigma)$ be the Banach space of totally measurable X^*-valued functions (i.e. uniform limits of simple functions) with the uniform norm. Since Y^* is isomorphic to the set of all f.a. X-measures on Σ having finite total variation [2], the Hahn-Banach theorem implies that for each $0 \neq g \in Y$ and $\alpha \in R$, there exists a f.a. X-measure μ such that $\int_S g d\mu = \alpha$, where the integral is defined in [3].

3. **The Lebesgue decomposition.** Definitions. Let β be an outer measure defined on Σ. Then $\mu: \Sigma \to X$ is *β-continuous* if $\beta(E) \to 0$ implies $\mu(E) \to 0$. μ is *β-singular* if there exists an $E^* \in \Sigma$ such that $\beta(E^*) = 0$ and $\mu(E) = \mu(E \cap E^*)$, $E \in \Sigma$.

THEOREM 2. *Let $\mu: \Sigma \to X$ be countably additive and let β be an outer measure defined on Σ. Then μ can be decomposed uniquely into the form $\mu = \mu_1 + \mu_2$, where μ_1 and μ_2 are countably additive, μ_1 is β-continuous, and μ_2 is β-singular.*

PROOF. The uniqueness of μ_1 and μ_2 is obvious. By a theorem due to Bartle, Dunford, and Schwartz [1], there exists a finite nonnegative c.a. measure λ defined on Σ such that μ is λ-continuous (see [5] for an elementary proof of this result). λ can be decomposed uniquely into a β-continuous part λ_c and a β-singular part λ_s. This can be seen by examining the proof of the classical Lebesgue decomposition theorem in [4, p. 132]. The argument used there remains valid if the appropriate measure appearing in the proof is replaced by an outer measure. Consequently, there exists a set $E^* \in \Sigma$ such that $\beta(E^*) = 0$ and $\lambda_s(E) = \lambda(E \cap E^*)$, $\lambda_c(E) = \lambda(E \cap (S - E^*))$. Let $\mu_1(E)$

$=\mu(E\cap(S-E^*))$ and $\mu_2(E)=\mu(E\cap E^*)$. Obviously μ_2 is β-singular. Now if $\beta(E_n)\to 0$, then $\lambda(E_n\cap(S-E^*))=\lambda_c(E_n)\to 0$; hence $\mu_1(E_n)=\mu(E_n\cap(S-E^*))\to 0$, and μ_1 is β-continuous. The conclusion of the theorem now follows.

ADDED IN PROOF. The classical Lebesgue decomposition theorem (even with respect to an outer measure) can be proved in a straightforward fashion that avoids the Radon-Nikodym theorem. This method was used by the author to decompose set functions of a more general type than the ones considered here (cf. *An integration theory for set-valued measures*. I, Bull. Soc. Roy. Sci. Liège, no5–8 (1968), 312–319). We shall sketch the proof. Let λ and β be as in the above proof. Define $\mathfrak{N}=\{E\in\Sigma:\beta(E)=0\}$; $\eta=\sup\lambda(E)$, $E\in\mathfrak{N}$. By using the method of exhaustion, one can construct a set E^* such that $E^*\in\mathfrak{N}$ and $\lambda(E^*)=\eta$. It follows that if $E\in\mathfrak{N}$, then $\lambda(E-E^*)=0$. Then define λ_s and λ_c to be the restrictions of λ to E^* and $S-E^*$ respectively.

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UNIVERSITY OF FLORIDA