DECOMPOSITION THEOREMS FOR VECTOR MEASURES

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1. Introduction. In this note we consider two theorems for vector measures, each of which is a generalization of a well-known theorem for scalar measures. Theorem 1 extends the Yosida-Hewitt theorem [7], [4, p. 163], which states that a bounded real valued finitely additive (f.a.) measure defined on a Boolean algebra can be decomposed uniquely into a countably additive (c.a.) part and a purely finitely additive (p.f.a.) part. The second theorem, due to Rickart [6, Theorem 4.5], is a Lebesgue decomposition theorem for c.a. vector measures with respect to an outer measure. Several authors have given alternate proofs of a restricted form of this theorem (e.g. see [3, p. 189] where the outer measure is replaced by a measure and the total variation of the vector measure is assumed to be finite—a condition not generally satisfied). In 3 we give a short and elementary proof of Rickart's theorem, which represents a considerable simplification of the existing proofs of this result.

Let \( X \) be a Banach space over the reals \( \mathbb{R} \) with first and second conjugate spaces \( X^* \) and \( X^{**} \); we regard \( X \) as a subset of \( X^{**} \). \( \Sigma_0 \) and \( \Sigma \) respectively denote a Boolean algebra and a \( \sigma \)-algebra of subsets of a set \( S \). \( \|\mu\| \) denotes the semivariation [3, p. 51] of \( \mu \). \( \mu \) is bounded if \( \|\mu\|(S) < \infty \).

2. Theorem 1. Let \( \mu: \Sigma_0 \rightarrow X \) be bounded and finitely additive. Then \( \mu \) can be written uniquely in the form \( \mu = \mu_1 + \mu_2 \), where the \( \mu_i: \Sigma_0 \rightarrow X^{**} \) are finitely additive and for each \( f \in X^* \): (1) \( \mu_1(\cdot) f: \Sigma_0 \rightarrow \mathbb{R} \) is countably additive and (2) \( \mu_2(\cdot) f: \Sigma_0 \rightarrow \mathbb{R} \) is purely finitely additive.

Proof. For \( f \in X^* \) the set function \( f\mu \) defined by \( (f\mu)(E) = f(\mu(E)) \), \( E \in \Sigma_0 \) is f.a. Moreover, since \( |f\mu(E)| \leq \|f\| \|\mu(E)\| \leq \|f\| \|\mu\|(S), f\mu \) is bounded. By the Yosida-Hewitt theorem \( f\mu = \mu_{f,1} + \mu_{f,2} \), where \( \mu_{f,1} \) is c.a. and \( \mu_{f,2} \) is p.f.a. For \( E \in \Sigma_0 \) define \( F_{i,B}: X^* \rightarrow \mathbb{R}, i = 1, 2 \), as follows: \( F_{i,B}(f) = \mu_{f,i}(E), f \in X^* \). It follows from the uniqueness of the decomposition of the scalar measures and Theorem 1.17 in [7] that \( \mu_{af,i} = a\mu_{f,i} \) and \( \mu_{f+g,i} = \mu_{f,i} + \mu_{g,i} \), for \( f, g \in X^*, a \in \mathbb{R} \); hence \( F_{i,B} \) is linear on \( X^* \). To show that \( F_{i,B} \in X^{**} \), let \( f \in X^* \) and let \( f\mu = (f\mu)^+ - (f\mu)^- \), \( \|f\mu\| = (f\mu)^+ + (f\mu)^- \) be the Jordan decomposition of \( f\mu \). Again using the uniqueness of the decomposition of \( f\mu \), we have \( u_{f,i} = (f\mu)^+_i - (f\mu)^-_i \) where the subscripts refer to the c.a. and p.f.a. parts.

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Define \( \gamma_i(E) = f_i, E \in \Sigma \). It is clear from the construction that \( \gamma_i \) and \( \gamma_2 \) have the required properties.

Corollary. Assume \( X \) is reflexive and \( \mu : \Sigma \rightarrow X \) is bounded and finitely additive. Then \( \mu \) can be written uniquely in the form \( \mu = \mu_1 + \mu_2 \), \( \mu_i : \Sigma \rightarrow X \), where \( (1') \mu_1 \) is countably additive and \( (2') f\mu_2 \) is purely finitely additive for each \( f \in X^* \).

Proof. Let \( \mu_i \) be the components of \( \mu \) given in the above theorem. Since \( f\mu_i() = \mu_i()f \) is c.a. for \( f \in X^* \), \( \mu_1 \) is weakly countably additive in addition to being f.a. By the Pettis theorem [4, p. 318] \( \mu_1 \) is c.a., and the result follows.

Remark. For a reflexive space \( X \), the existence of a large number of bounded f.a. \( X \)-valued measures on \( \Sigma \), hence measures satisfying \( (2') \), can be deduced as follows. Let \( Y = M_{X^*}(\Sigma) \) be the Banach space of totally measurable \( X^* \)-valued functions (i.e. uniform limits of simple functions) with the uniform norm. Since \( Y^* \) is isomorphic to the set of all f.a. \( X \)-measures on \( \Sigma \) having finite total variation [2], the Hahn-Banach theorem implies that for each \( 0 \neq g \in Y \) and \( \alpha \in \mathbb{R} \), there exists a f.a. \( X \)-measure \( \mu \) such that \( \int g d\mu = \alpha \), where the integral is defined in [3].

3. The Lebesgue decomposition. Definitions. Let \( \beta \) be an outer measure defined on \( \Sigma \). Then \( \mu : \Sigma \rightarrow X \) is \( \beta \)-continuous if \( \beta(E) \rightarrow 0 \) implies \( \mu(E) \rightarrow 0 \). \( \mu \) is \( \beta \)-singular if there exists an \( E^* \in \Sigma \) such that \( \beta(E^*) = 0 \) and \( \mu(E) = \mu(E \cap E^*) \), \( E \in \Sigma \).

Theorem 2. Let \( \mu : \Sigma \rightarrow X \) be countably additive and let \( \beta \) be an outer measure defined on \( \Sigma \). Then \( \mu \) can be decomposed uniquely into the form \( \mu = \mu_1 + \mu_2 \), where \( \mu_1 \) and \( \mu_2 \) are countably additive, \( \mu_1 \) is \( \beta \)-continuous, and \( \mu_2 \) is \( \beta \)-singular.

Proof. The uniqueness of \( \mu_1 \) and \( \mu_2 \) is obvious. By a theorem due to Bartle, Dunford, and Schwartz [1], there exists a finite nonnegative c.a. measure \( \lambda \) defined on \( \Sigma \) such that \( \mu = \lambda \)-continuous (see [5] for an elementary proof of this result). \( \lambda \) can be decomposed uniquely into a \( \beta \)-continuous part \( \lambda_1 \) and a \( \beta \)-singular part \( \lambda_2 \). This can be seen by examining the proof of the classical Lebesgue decomposition theorem in [4, p. 132]. The argument used there remains valid if the appropriate measure appearing in the proof is replaced by an outer measure. Consequently, there exists a set \( E^* \in \Sigma \) such that \( \beta(E^*) = 0 \) and \( \lambda_1(E) = \lambda(E \cap E^*), \lambda_2(E) = \lambda(E \cap (S - E^*)) \). Let \( \mu_1(E) \)
= \mu(E \cap (S - E^*)) and \mu_2(E) = \mu(E \cap E^*). Obviously \mu_2 is \beta\text{-singular. Now if } \beta(E_n) \to 0, then \lambda(E_n \cap (S - E^*)) = \lambda_2(E_n) \to 0; hence \mu_1(E_n) = \mu(E_n \cap (S - E^*)) \to 0, and \mu_1 is \beta\text{-continuous. The conclusion of the theorem now follows.

**Added in proof.** The classical Lebesgue decomposition theorem (even with respect to an outer measure) can be proved in a straightforward fashion that avoids the Radon-Nikodym theorem. This method was used by the author to decompose set functions of a more general type than the ones considered here (cf. *An integration theory for set-valued measures*. I, Bull. Soc. Roy. Sci. Liège, no 5-8 (1968), 312–319). We shall sketch the proof. Let \lambda and \beta be as in the above proof. Define \mathcal{R} = \{E \in \Sigma: \beta(E) = 0\}; \eta = \sup \lambda(E), E \in \mathcal{R}. By using the method of exhaustion, one can construct a set \mathcal{E}^* such that \mathcal{E}^* \subseteq \mathcal{R} and \lambda(\mathcal{E}^*) = \eta. It follows that if \mathcal{E} \in \mathcal{R}, then \lambda(\mathcal{E} - \mathcal{E}^*) = 0. Then define \lambda_1 and \lambda_2 to be the restrictions of \lambda to \mathcal{E}^* and \mathcal{E} - \mathcal{E}^* respectively.

**Bibliography**


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