DECOMPOSITION THEOREMS FOR VECTOR MEASURES

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1. Introduction. In this note we consider two theorems for vector measures, each of which is a generalization of a well-known theorem for scalar measures. Theorem 1 extends the Yosida-Hewitt theorem [7], [4, p. 163], which states that a bounded real valued finitely additive (f.a.) measure defined on a Boolean algebra can be decomposed uniquely into a countably additive (c.a.) part and a purely finitely additive (p.f.a.) part. The second theorem, due to Rickart [6, Theorem 4.5], is a Lebesgue decomposition theorem for c.a. vector measures with respect to an outer measure. Several authors have given alternate proofs of a restricted form of this theorem (e.g. see [3, p. 189] where the outer measure is replaced by a measure and the total variation of the vector measure is assumed to be finite—a condition not generally satisfied). In 3 we give a short and elementary proof of Rickart's theorem, which represents a considerable simplification of the existing proofs of this result.

Let $X$ be a Banach space over the reals $\mathbb{R}$ with first and second conjugate spaces $X^*$ and $X^{**}$; we regard $X$ as a subset of $X^{**}$. $\Sigma_0$ and $\Sigma$ respectively denote a Boolean algebra and a $\sigma$-algebra of subsets of a set $S$. $\mu$ denotes the semivariation [3, p. 51] of $\mu$. $\mu$ is bounded if $\|\mu\|(S) < \infty$.

2. Theorem 1. Let $\mu: \Sigma_0 \rightarrow X$ be bounded and finitely additive. Then $\mu$ can be written uniquely in the form $\mu = \mu_1 + \mu_2$, where the $\mu_i: \Sigma_0 \rightarrow X$ are finitely additive and for each $f \in X^*$: (1) $\mu_1(f): \Sigma_0 \rightarrow \mathbb{R}$ is countably additive and (2) $\mu_2(f): \Sigma_0 \rightarrow \mathbb{R}$ is purely finitely additive.

Proof. For $f \in X^*$ the set function $f\mu$ defined by $(f\mu)(E) = f(\mu(E))$, $E \in \Sigma_0$ is f.a. Moreover, since $|f\mu(E)| \leq \|f\| \|\mu(E)\| \leq \|f\| \|\mu\|(S)$, $f\mu$ is bounded. By the Yosida-Hewitt theorem $f\mu = \mu_{i1} + \mu_{i2}$, where $\mu_{i1}$ is c.a. and $\mu_{i2}$ is p.f.a. For $E \in \Sigma_0$ define $F_{i,B}: X^* \rightarrow \mathbb{R}$, $i = 1, 2$, as follows: $F_{i,B}(f) = \mu_{i,B}(E)$, $f \in X^*$. It follows from the uniqueness of the decomposition of the scalar measures and Theorem 1.17 in [7] that $\mu_{i,f} = \alpha \mu_{i1}$ and $\mu_{i,f+g} = \mu_{i1} + \mu_{i2}$, for $f, g \in X^*$, $\alpha \in \mathbb{R}$; hence $F_{i,B}$ is linear on $X^*$. To show that $F_{i,B} \in X^{**}$, let $f \in X^*$ and let $f\mu = (f\mu)^+ - (f\mu)^-$, $|f\mu| = (f\mu)^+ + (f\mu)^-$ be the Jordan decomposition of $f\mu$. Again using the uniqueness of the decomposition of $f\mu$, we have $u_{f,i} = (f\mu)^+ - (f\mu)^-$, where the subscripts refer to the c.a. and p.f.a. parts. $|F_{i,B}(f)|$
\[ |\mu_f(E) - (f\mu)_f(E)| \leq (f\mu)(E) + (f\mu)_f(E) \leq (f\mu)(E) + (f\mu)(E) = |f\mu(E)| \leq 2 \sup_{A \in E} |f\mu(A)| \leq 2 \sup_{A \in E} \|f\| \|\mu(A)\| \leq 2 \|f\| \|\mu\|(E). \] Thus \(\|F_{1, E}\| \leq 2 \|\mu\|(E)\) and \(F_{1, E} \in X^{**}\). Define \(\mu_1(E) = F_{1, E}\), \(E \in \Sigma\). It is clear from the construction that \(\mu_1\) and \(\mu_2\) have the required properties.

**Corollary.** Assume \(X\) is reflexive and \(\mu: \Sigma \to X\) is bounded and finitely additive. Then \(\mu\) can be written uniquely in the form \(\mu = \mu_1 + \mu_2\), \(\mu_i: \Sigma \to X\), where (1') \(\mu_1\) is countably additive and (2') \(f\mu_2\) is purely finitely additive for each \(f \in X^*\).

**Proof.** Let \(\mu_i\) be the components of \(\mu\) given in the above theorem. Since \(f\mu_1(\cdot) = \mu_1(\cdot)f\) is c.a. for \(f \in X^*\), \(\mu_1\) is weakly countably additive in addition to being f.a. By the Pettis theorem [4, p. 318] \(\mu_1\) is c.a., and the result follows.

**Remark.** For a reflexive space \(X\), the existence of a large number of bounded f.a. \(X\)-valued measures on \(\Sigma\), hence measures satisfying (2'), can be deduced as follows. Let \(Y = \mathcal{M}_X^*(\Sigma)\) be the Banach space of totally measurable \(X^*\)-valued functions (i.e. uniform limits of simple functions) with the uniform norm. Since \(Y^*\) is isomorphic to the set of all f.a. \(X\)-measures on \(\Sigma\) having finite total variation [2], the Hahn-Banach theorem implies that for each \(0 \neq g \in Y\) and \(\alpha \in \mathbb{R}\), there exists a f.a. \(X\)-measure \(\mu\) such that \(\int g d\mu = \alpha\), where the integral is defined in [3].

### 3. The Lebesgue decomposition.

**Definitions.** Let \(\beta\) be an outer measure defined on \(\Sigma\). Then \(\mu: \Sigma \to X\) is \(\beta\)-continuous if \(\beta(E) \to 0\) implies \(\mu(E) \to 0\). \(\mu\) is \(\beta\)-singular if there exists an \(E^* \in \Sigma\) such that \(\beta(E^*) = 0\) and \(\mu(E) = \mu(E \cap E^*)\), \(E \in \Sigma\).

**Theorem 2.** Let \(\mu: \Sigma \to X\) be countably additive and let \(\beta\) be an outer measure defined on \(\Sigma\). Then \(\mu\) can be decomposed uniquely into the form \(\mu = \mu_1 + \mu_2\), where \(\mu_1\) and \(\mu_2\) are countably additive, \(\mu_1\) is \(\beta\)-continuous, and \(\mu_2\) is \(\beta\)-singular.

**Proof.** The uniqueness of \(\mu_1\) and \(\mu_2\) is obvious. By a theorem due to Bartle, Dunford, and Schwartz [1], there exists a finite nonnegative c.a. measure \(\lambda\) defined on \(\Sigma\) such that \(\mu\) is \(\lambda\)-continuous (see [5] for an elementary proof of this result). \(\lambda\) can be decomposed uniquely into a \(\beta\)-continuous part \(\lambda_c\) and a \(\beta\)-singular part \(\lambda_s\). This can be seen by examining the proof of the classical Lebesgue decomposition theorem in [4, p. 132]. The argument used there remains valid if the appropriate measure appearing in the proof is replaced by an outer measure. Consequently, there exists a set \(E^* \in \Sigma\) such that \(\beta(E^*) = 0\) and

\[ \lambda_c(E) = \lambda(E \cap E^*), \quad \lambda_s(E) = \lambda(E \cap (S - E^*)). \]

Let \(\mu_1(E)\)
= \mu(E \cap (S - E^*)) and \mu_2(E) = \mu(E \cap E^*). Obviously \mu_2 is \beta\text{-singular. Now if } \beta(E_n) \to 0, then \lambda(E_n \cap (S - E^*)) = \lambda_\nu(E_n) \to 0; hence \mu_1(E_n) = \mu(E_n \cap (S - E^*)) \to 0, and \mu_1 is \beta\text{-continuous. The conclusion of the theorem now follows.}

**Added in proof.** The classical Lebesgue decomposition theorem (even with respect to an outer measure) can be proved in a straightforward fashion that avoids the Radon-Nikodym theorem. This method was used by the author to decompose set functions of a more general type than the ones considered here (cf. *An integration theory for set-valued measures*. I, Bull. Soc. Roy. Sci. Liège, no5–8 (1968), 312–319). We shall sketch the proof. Let \( \lambda \) and \( \beta \) be as in the above proof. Define \( \mathcal{N} = \{ E \in \Sigma: \beta(E) = 0 \} \); \( \eta = \sup \lambda(E), E \in \mathcal{N}. \) By using the method of exhaustion, one can construct a set \( E^* \) such that \( E^* \in \mathcal{N} \) and \( \lambda(E^*) = \eta. \) It follows that if \( E \in \mathcal{N}, \) then \( \lambda(E - E^*) = 0. \) Then define \( \lambda_\nu \) and \( \lambda_\tau \) to be the restrictions of \( \lambda \) to \( E^* \) and \( S - E^* \) respectively.

**Bibliography**


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