Suppose $U$ is the unit disc in $\mathbb{C}$. For $0 < r < 1$ $Q_r$ (or simply $Q$) is the annulus $\{z \in U \mid |z| > r\}$. A subvariety $V$ of pure codimension 1 in $U^N$ is called a Rudin subvariety if for some $r$ $V \cap Q^r = \emptyset$. A Rudin subvariety is called a special Rudin subvariety if there is $\delta > 0$ such that, for $1 \leq k \leq N$, $(a', a_i, a'' \in (Q^{k-1} \times U \times Q^{N-k}) \cap V$, $i = 1, 2$, and $a_i \neq a'_i$, we have $|a_i - a'_i| \geq \delta$. If a holomorphic function $f$ generates the ideal-sheaf of its zero-set $E$, then we write $Z(f) = E$. The Banach space of all bounded holomorphic functions on a reduced complex space $X$ under the sup norm is denoted by $H^\infty(X)$ and the norm of $f \in H^\infty(X)$ is denoted by $\|f\|_{\infty}$. The following two theorems were proved by W. Rudin [2] and H. Alexander [1] respectively.

**Theorem 1.** If $V$ is a Rudin subvariety, then there is $f \in H^\infty(U^N)$ such that $Z(f) = V$.

**Theorem 2.** If $V$ is a special Rudin subvariety, then there is a bounded linear map from $H^\infty(V)$ to $H^\infty(U^N)$ which extends every bounded holomorphic function on $V$ to one on $U^N$.

Cartan’s Theorem B implies that an analytic hypersurface of a polydisc is the zero-set of a holomorphic function and that every holomorphic function on the hypersurface is induced by a holomorphic function on the polydisc. One can expect that some Theorem B with bounds would easily yield the above two theorems. In this note we prove a simple theorem on sheaf cohomology with bounds (Theorem 3 below) which can imply Theorems 1 and 2. This gives us more perspective proofs of these two theorems.

Suppose $X$ is a reduced complex space and $\mathcal{O}$ is the structure-sheaf of $X \times U^N$. Let $W_k = X \times U^{k-1} \times Q \times U^{N-k}$, $1 \leq k \leq N$, and $\mathcal{W} = \{W_k\}$. For $\nu \geq 0$ and $1 \leq i_0, \ldots, i_v \leq N$, $W_{i_0, \ldots, i_v}$ denotes $W_{i_0} \cap \cdots \cap W_{i_v}$. If $f \in \mathcal{C}(\mathcal{W}, \mathcal{O})$, then $f_{i_0, \ldots, i_v} \in \Gamma(W_{i_0, \ldots, i_v}, \mathcal{O})$ denotes the value of $f$ at the simplex $(W_{i_0, \ldots, i_v})$ of the nerve of $\mathcal{W}$. Let $\rho = 2/(1 - r)$ and for $1 \leq \nu < N$ let

$$\sigma_\nu = \sum_{\mu=1}^{N} \binom{N}{\mu} (\nu + 1)^{\mu-1} \rho^\mu.$$
Lemma 1. Suppose $f$ is a bounded holomorphic function on $X \times Q$ whose absolute value is bounded by a positive number $K$. Suppose for $w \in X$, $f(w, z) = \sum_{\nu=0}^{\infty} h_{\nu}(w)z^{\nu}$ is the Laurent series expansion of $f$ in $z$ (where $z$ is the coordinate function of $Q$). Let $g(w, z) = \sum_{\nu=0}^{\infty} h_{\nu}(w)z^{\nu}$ on $X \times Q$. Then $\|g\|_{X \times Q} \leq pK$.

Proof. Fix $(w, z) \in X \times Q$. Choose arbitrarily two positive numbers $a$ and $b$ such that $r < a < |z| < b < 1$. We need only prove that $|g(w, z)| \leq 2bK/(b-a)$, because the result follows then from letting $a \to r$ and $b \to 1$.

Case (i). $|z| \leq (a+b)/2$. Then $|\zeta - z| \leq (b-a)/2$ for $|\zeta| = b$.

$$|g(w, z)| = \left| \frac{1}{2\pi i} \int_{|\zeta| = b} \frac{f(w, \zeta)}{\zeta - z} d\zeta \right| \leq \frac{2b}{b-a} K.$$ 

Case (ii). $|z| \leq (a+b)/2$. Then $|\zeta - z| \leq (b-a)/2$ for $|\zeta| = a$.

$$|f(w, z) - g(w, z)| = \left| \frac{-1}{2\pi i} \int_{|\zeta| = a} \frac{f(w, \zeta)}{\zeta - z} d\zeta \right| \leq \frac{2a}{b-a} K.$$

Hence $|g(w, z)| \leq 2bK/(b-a)$. Q.E.D.

Theorem 3. For $1 \leq \nu < N$ there exists a linear map $\phi_{\nu}: B^{r}(\mathbb{R}, \mathfrak{o}) \to C^{r-1}(\mathbb{R}, \mathfrak{o})$ over the ring of all holomorphic functions on $X$ such that

(i) $\delta \phi_{\nu} = \text{the identity map on } B^{r}(\mathbb{R}, \mathfrak{o})$, and

(ii) if $f \in B^{r}(\mathbb{R}, \mathfrak{o})$ and $\|f_{i_0, \ldots, i_N}\|_{W_{i_0, \ldots, i_N}} \leq K$ for $1 \leq i_0, \ldots, i_N \leq N$, then $\|\phi_{\nu}(f)\|_{W_{i_0, \ldots, i_{N-1}}} \leq \sigma K$ for $1 \leq i_0, \ldots, i_{N-1} \leq N$.

Proof. First we define for $1 \leq i \leq N$ and $0 \leq \nu < N$ a linear map $e_{i}: C^{r}(\mathbb{R}, \mathfrak{o}) \to C^{r}(\mathbb{R}, \mathfrak{o})$ over the ring of all holomorphic functions on $X$ as follows: Suppose $f \in C^{r}(\mathbb{R}, \mathfrak{o})$. If $f_{i_0, \ldots, i_N} = \sum_{\mu=0}^{N} h_{\mu}^{(i_0, \ldots, i_N)} e^{\mu}_{i_0} \ldots e^{(i_N)}$, then $e_{i}(f)_{i_0, \ldots, i_N} = \sum_{\mu=0}^{N} h_{\mu}^{(i_0, \ldots, i_N)} e^{\mu}_{i}$. By applying Lemma 1 with $X$ replaced by the product of $X$ and $U_{i_0, \ldots, i_N}$, we have $\|e_{i}(f)_{i_0, \ldots, i_N}\|_{W_{i_0, \ldots, i_N}} \leq \rho \|f_{i_0, \ldots, i_N}\|_{W_{i_0, \ldots, i_N}}$. Observe that $((1-e_{i_0})(f)_{i_0, \ldots, i_N} = 0$ if $i \neq i_0, \ldots, i_N$. For $0 \leq \nu < N-1$ we have $(1-e_{i_0}) \circ (1-e_{i_2}) \circ \cdots \circ (1-e_{i_N}) = 0$ on $C^{r}(\mathbb{R}, \mathfrak{o})$, because for any $1 \leq i_0, \ldots, i_N \leq N$ there exists $1 \leq i \leq N$ such that $i \neq i_0, \ldots, i_N$. Since $e_{i}$ commutes with $\delta$, for $1 \leq i < N$ we have $(1-e_{i}) \circ (1-e_{i_2}) \circ \cdots \circ (1-e_{i_N}) = 0$ on $B^{r}(\mathbb{R}, \mathfrak{o})$.

Next we define for $1 \leq i \leq N$ and $1 \leq \nu < N$ a linear map $k_{i}: C^{r}(\mathbb{R}, \mathfrak{o}) \to C^{r-1}(\mathbb{R}, \mathfrak{o})$ over the ring of all holomorphic functions on $X$ as follows: If $f \in C^{r}(\mathbb{R}, \mathfrak{o})$, then set $(k_{i}(f))_{i_0, \ldots, i_{N-1}}$ to be the holomorphic function on $W_{i_0, \ldots, i_{N-1}}$ whose restriction to $W_{i_0, \ldots, i_{N-1}}$ is $(e_{i}(f))_{i_0, \ldots, i_{N-1}}$. Straightforward computation shows that for $1 \leq i \leq N$ and $1 \leq \nu < N$ we have $e_{i} = \delta k_{i} - k_{ii}$ on $C^{r}(\mathbb{R}, \mathfrak{o})$. Hence for $1 \leq \nu < N$ we have $(1-\delta k_{i}) \circ (1-\delta k_{2}) \circ \cdots \circ (1-\delta k_{N}) = 0$ on $B^{r}(\mathbb{R}, \mathfrak{o})$. For $1 \leq \nu < N$
define \( \phi_r : B^r(\mathbb{R}, 0) \to C^{-1}(\mathbb{R}, 0) \) by
\[
\phi_r = \sum_{\mu=1}^{N} (-1)^{n-1} \sum_{i_1 < \ldots < i_n} k_{i_1} \delta k_{i_2} \cdots \delta k_{i_n}.
\]
Then \( \phi_r \) satisfies the requirement. Q.E.D.

Remark. By using \( \sup |\text{Re} f_{i_1} \cdots f_{i_n}| \) on \( W_{i_1} \cdots i_{n-1} \) instead of using \( \|f_{i_1} \cdots f_{i_n}\|_{W_{i_1} \cdots i_{n-1}} \) and \( \|\phi_r(f_{i_1} \cdots f_{i_n})\|_{W_{i_1} \cdots i_{n-1}} \), a theorem similar to Theorem 3 can be proved. We need only prove a lemma which corresponds to Lemma 1 but uses sup norms of the real parts instead. To do this, we observe that \( f \mapsto \text{Re} f \) defines a continuous \( \mathcal{R} \)-linear injection with closed image from the Fréchet space \( E \) of all holomorphic functions on \( Q \) whose constant coefficients in the Laurent series expansions are real to the Fréchet space of all harmonic functions on \( Q \). Hence, for \( r < a < b < 1 \), there exists a constant \( C \) such that, if \( f \in E \) and \( \sup |\text{Re} f| \leq K \), then \( |f(z)| \leq CK \) on \( a \leq |z| \leq b \). The desired lemma follows from an argument analogous to the proof of Lemma 1, but this time we leave \( a \) and \( b \) fixed instead of letting \( a \to r \) and \( b \to 1 \) and do not restrict \( |z| \) to \((a, b)\).

Proof of Theorem 1. By Cartan's Theorem B there is a holomorphic function \( \tilde{f} \) on \( U^N \) such that \( Z(\tilde{f}) = V \). We can assume \( V \cap (Q_{r'})^N = \emptyset \) for some \( r' < r \). We are going to prove \((1)^*\) by induction on \( k \).

On \( U^k \times Q^{N-k} \) (and likewise on products obtained by permuting the \( N \) factors) we can construct a bounded holomorphic function \( f^{(k)}_1 \) such that \( Z(f^{(k)}_1) = (U^k \times Q^{N-k}) \cap V \) and \( (f^{(k)}_1)^{-1} \) is bounded on \( Q^N \).

Complements on \( Q^N \cap V = \emptyset \) implies that \((U \times Q^{N-1}) \cap V \) is an analytic cover over \( Q^{N-1} \) of, say, \( n \) sheets. There exists a proper subvariety \( A \) in \( Q^{N-1} \) and locally defined holomorphic functions \( g^{(1)}, \ldots, g^{(n)} \) on \( Q^{N-1} \) such that \( (U \times (Q^{N-1} - A)) \cap V = \{ (z_1, \ldots, z_N) \in U \times (Q^{N-1} - A) \mid z_i = g^{(i)}(z_2, \ldots, z_N) \text{ for some } i \} \). The bounded holomorphic extension \( f^{(1)}_1 \) on \( U \times Q^{N-1} \) of \( \prod_{i=1}^{n} (z_i - g^{(i)}(z_2, \ldots, z_N)) \) satisfies \( Z(f^{(1)}_1) = (U \times Q^{N-1}) \cap V \) and \( (f^{(1)}_1)^{-1} \) is bounded on \( Q^N \). (1) is proved. Suppose \( (1)_k \) is true for \( 1 \leq k < m \). Then for \( 1 \leq i \leq m \) we can construct a bounded holomorphic function \( f_i \) on \( G_i = U^{i-1} \times Q \times U^{m-i} \times Q^{N-m} \) such that \( Z(f_i) = G_i \cap V \) and \( f_i^{-1} \) is bounded on \( Q^N \). By replacing \( f_i \) by the product of \( f_i \) with suitable powers of \( z_i, z_{m+i}, \ldots, z_N \), we can assume that we can select a regular branch \( h_i \) of \( \log(f_i f_i^{-1}) \) on \( G_i \). Since \( h_i - h_j = \text{a branch of } \log(f_j f_i^{-1}) \) has bounded real part on \( G_i \cap G_j \), by the Remark following Theorem 3 we can construct holomorphic functions
\( h_i \) on \( G_i \) with bounded real parts such that \( h_i - h_j = a \) branch of \( \log(f_i/f_j) \). The holomorphic function \( f^{(m)} \) on \( U^m \times Q^{N-m} \) which agrees with \( f_i \exp(h_i) \) on \( G_i \) satisfies \( Z(f^{(m)}) = (U^m \times Q^{N-m}) \cap V \) and is bounded. Moreover, \( (f^{(m)})^{-1} \) is bounded on \( Q^N \). (1)\( m \) is proved. The theorem follows from (1)\( n \).

**Proof of Theorem 2.** By Theorem 1 we can construct \( g \in H^\infty(U^N) \) such that \( Z(g) = V \). The construction implies that \( g^{-1} \) is bounded on \( Q^N \). Take \( f \in H^\infty(V) \). By Cartan's Theorem B if is the restriction to \( V \) of a holomorphic function \( f \) on \( U^N \). We are going to prove (2)\( k \) by induction on \( k \).

On \( U^k \times Q^{N-k} \) (and likewise on products obtained by permuting the \( N \) factors) we can construct a bounded holomorphic function \( f^{(k)} \) which agrees with \( f \) on \( (U^k \times Q^{N-k}) \cap V \).

From the conditions of special Rudin subvarieties we conclude that \((U^k \times Q^{N-k}) \cap V \) is an unbranched analytic cover over \( Q^{N-1} \) of, say, \( n \) sheets. There are locally defined holomorphic functions \( g^{(i)} \) such that \( (U^k \times Q^{N-k}) \cap V = \{ (z_1, \ldots, z_N) \in U^k \times Q^{N-k} \mid z_1 = g^{(i)}(z_2, \ldots, z_N) \} \) for some \( i \). The function \( f^{(i)} = \sum_{i=1}^n f^{(i)}(g^{(i)}(z_2, \ldots, z_N), z_2, \ldots, z_N) g^{(i)}(z_2, \ldots, z_N) g^{(j)}(z_2, \ldots, z_N) \) is well defined, agrees with \( f \) on \( (U^k \times Q^{N-k}) \cap V \), and is bounded. (2)\( i \) is proved. Suppose (2)\( k \) is true for \( 1 \leq k < m \). We can construct bounded holomorphic functions \( f_i \) on \( G_i = U^{i-1} \times Q \times U^{m-i} \times Q^{N-m} \), \( 1 \leq i \leq m \), such that \( f_i = f \) on \( G_i \cap V \). Let \( h_i = (f_i - f_i)/g \) on \( G_i \). Since \( h_i - h_i = (f_i - f_i)/g \) is bounded on \( G_i \cap G_i \), we can construct by Theorem 3 \( h_i \in H^\infty(G_i) \) such that \( h_i - h_i = h_i - h_i = (f_i - f_i)/g \). The holomorphic function \( f^{(m)} \) on \( U^m \times Q^{N-m} \) which agrees with \( f_i + gh_i \) on \( G_i \), is bounded and agrees with \( f \) on \( (U^m \times Q^{N-m}) \cap V \). (2)\( m \) is proved. By (2)\( n \) we can construct \( f^{(N)} \in H^\infty(U^N) \) which agrees with \( f \) on \( V \). It is clear from the constructions that the map defined by \( f \mapsto f^{(N)} \) is a bounded linear map from \( H^\infty(V) \) to \( H^\infty(U^N) \).

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**References**


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