SHEAF COHOMOLOGY WITH BOUNDS AND BOUNDED HOLOMORPHIC FUNCTIONS

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Suppose U is the unit disc in $\mathbf{C}$. For $0<r<1$ Q_r (or simply Q) is the annulus $\{z\in U \mid |z|>r\}$. A subvariety V of pure codimension 1 in U^N is called a *Rudin subvariety* if for some r $V\cap Q^N=\emptyset$. A Rudin subvariety is called a *special Rudin subvariety* if there is $\delta>0$ such that, for $1\leqq k\leqq N$, $(z', a_i, z'')\in(Q^{k-1}\times U\times Q^{N-k})\cap V$, $i=1, 2$, and $a_1\neq a_2$, we have $|a_1-a_2|\geqq\delta$. If a holomorphic function f generates the ideal-sheaf of its zero-set E, then we write $Z(f)=E$. The Banach space of all bounded holomorphic functions on a reduced complex space X under the sup norm is denoted by $H^\infty(X)$ and the norm of $f\in H^\infty(X)$ is denoted by $\|f\|_X$. The following two theorems were proved by W. Rudin [2] and H. Alexander [1] respectively.

THEOREM 1. *If V is a Rudin subvariety, then there is $f\in H^\infty(U^N)$ such that $Z(f)=V$.*

THEOREM 2. *If V is a special Rudin subvariety, then there is a bounded linear map from $H^\infty(V)$ to $H^\infty(U^N)$ which extends every bounded holomorphic function on V to one on U^N.*

Cartan's Theorem B implies that an analytic hypersurface of a polydisc is the zero-set of a holomorphic function and that every holomorphic function on the hypersurface is induced by a holomorphic function on the polydisc. One can expect that some Theorem B with bounds would easily yield the above two theorems. In this note we prove a simple theorem on sheaf cohomology with bounds (Theorem 3 below) which can imply Theorems 1 and 2. This gives us more perspective proofs of these two theorems.

Suppose X is a reduced complex space and $\mathcal{O}$ is the structure-sheaf of $X\times U^N$. Let $W_k=X\times U^{k-1}\times Q\times U^{N-k}$, $1\leqq k\leqq N$, and $\mathfrak{W}=\{W_k\}$. For $\nu\geqq 0$ and $1\leqq i_0, \cdots, i_\nu\leqq N$ $W_{i_0\cdots i_\nu}$ denotes $W_{i_0}\cap\cdots\cap W_{i_\nu}$. If $f\in C^\nu(\mathfrak{W}, \mathcal{O})$, then $f_{i_0\cdots i_\nu}\in\Gamma(W_{i_0\cdots i_\nu}, \mathcal{O})$ denotes the value of f at the simplex $(W_{i_0}, \cdots, W_{i_\nu})$ of the nerve of $\mathfrak{W}$. Let $\rho=2/(1-r)$ and for $1\leqq\nu<N$ let

$$\sigma_\nu=\sum_{\mu=1}^{N}\binom{N}{\mu}(\nu+1)^{\mu-1}\rho^\mu.$$

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LEMMA 1. *Suppose f is a bounded holomorphic function on $X\times Q$ whose absolute value is bounded by a positive number K. Suppose for $w\in X$ $f(w, z)=\sum_{\mu=-\infty}^{\infty} h_\mu(w)z^\mu$ is the Laurent series expansion of f in z (where z is the coordinate function of Q). Let $g(w, z)=\sum_{\mu=0}^{\infty} h_\mu(w)z^\mu$ on $X\times Q$. Then $\|g\|_{X\times Q}\leqq \rho K$.*

PROOF. Fix $(w, z)\in X\times Q$. Choose arbitrarily two positive numbers a and b such that $r<a<|z|<b<1$. We need only prove that $|g(w, z)| \leqq 2bK/(b-a)$, because the result follows then from letting $a\to r$ and $b\to 1$.

Case (i). $|z|\leqq(a+b)/2$. Then $|\zeta-z|\geqq(b-a)/2$ for $|\zeta|=b$.

$$|g(w, z)| = \left|\frac{1}{2\pi i}\int_{|\zeta|=b}\frac{f(w,\zeta)}{\zeta - z}\,d\zeta\right| \leqq \frac{2b}{b-a}K.$$

Case (ii). $|z|\geqq(a+b)/2$. Then $|\zeta-z|\geqq(b-a)/2$ for $|\zeta|=a$.

$$|f(w, z) - g(w, z)| = \left|\frac{-1}{2\pi i}\int_{|\zeta|=a}\frac{f(w,\zeta)}{\zeta - z}\,d\zeta\right| \leqq \frac{2a}{b-a}K.$$

Hence $|g(w, z)|\leqq 2bK/(b-a)$. Q.E.D.

THEOREM 3. *For $1\leqq\nu<N$ there exists a linear map ϕ_ν: $B^\nu(\mathfrak{W}, \mathcal{O})\to C^{\nu-1}(\mathfrak{W}, \mathcal{O})$ over the ring of all holomorphic functions on X such that*

(i) *$\delta\phi_\nu=$ the identity map on $B^\nu(\mathfrak{W}, \mathcal{O})$, and*

(ii) *if $f\in B^\nu(\mathfrak{W}, \mathcal{O})$ and $\|f_{i_0\cdots i_\nu}\|_{W_{i_0\cdots i_\nu}}\leqq K$ for $1\leqq i_0, \cdots, i_\nu\leqq N$, then $\|(\phi_\nu(f))_{i_0\cdots i_{\nu-1}}\|_{W_{i_0\cdots i_{\nu-1}}}\leqq\sigma_\nu K$ for $1\leqq i_0, \cdots, i_{\nu-1}\leqq N$.*

PROOF. First we define for $1\leqq i\leqq N$ and $0\leqq\nu<N$ a linear map e_i: $C^\nu(\mathfrak{W}, \mathcal{O})\to C^\nu(\mathfrak{W}, \mathcal{O})$ over the ring of all holomorphic functions on X as follows: Suppose $f\in C^\nu(\mathfrak{W}, \mathcal{O})$. If $f_{i_0\cdots i_\nu}=\sum_{\mu=-\infty}^{\infty} h_\mu^{(i_0,\cdots,i_\nu)}z_i^\mu$ is the Laurent series expansion of $f_{i_0\cdots i_\nu}$ in z_i (where z_i is the ith coordinate function of U^N), then $(e_i(f))_{i_0\cdots i_\nu}=\sum_{\mu=0}^{\infty} h_\mu^{(i_0,\cdots,i_\nu)}z_i^\mu$. By applying Lemma 1 with X replaced by the product of X and U^{N-1}, we have $\|(e_i(f))_{i_0\cdots i_\nu}\|_{W_{i_0\cdots i_\nu}}\leqq\rho\|f_{i_0\cdots i_\nu}\|_{W_{i_0\cdots i_\nu}}$. Observe that $((1-e_i)(f))_{i_0\cdots i_\nu}=0$ if $i\neq i_0, \cdots, i_\nu$. For $0\leqq\nu<N-1$ we have $(1-e_1)\circ(1-e_2)\circ\cdots\circ(1-e_N)=0$ on $C^\nu(\mathfrak{W}, \mathcal{O})$, because for any $1\leqq i_0, \cdots, i_\nu\leqq N$ there exists $1\leqq i\leqq N$ such that $i\neq i_0, \cdots, i_\nu$. Since e_i commutes with δ, for $1\leqq\nu<N$ we have $(1-e_1)\circ(1-e_2)\circ\cdots\circ(1-e_N)=0$ on $B^\nu(\mathfrak{W}, \mathcal{O})$.

Next we define for $1\leqq i\leqq N$ and $1\leqq\nu<N$ a linear map k_i: $C^\nu(\mathfrak{W}, \mathcal{O})\to C^{\nu-1}(\mathfrak{W}, \mathcal{O})$ over the ring of all holomorphic functions on X as follows: If $f\in C^\nu(\mathfrak{W}, \mathcal{O})$, then set $(k_i(f))_{i_0\cdots i_{\nu-1}}$ to be the holomorphic function on $W_{i_0\cdots i_{\nu-1}}$ whose restriction to $W_{i\,i_{\nu-1}}$ is $(e_i(f))_{i\,i_{\nu-1}}$. Straightforward computation shows that for $1\leqq i\leqq N$ and $1\leqq\nu<N$ we have $e_i=\delta k_i-k_i\delta$ on $C^\nu(\mathfrak{W}, \mathcal{O})$. Hence for $1\leqq\nu<N$ we have $(1-\delta k_1)\circ(1-\delta k_2)\circ\cdots\circ(1-\delta k_N)=0$ on $B^\nu(\mathfrak{W}, \mathcal{O})$. For $1\leqq\nu<N$

define ϕ_ν: $B^\nu(\mathfrak{W}, \mathcal{O}) \to C^{\nu-1}(\mathfrak{W}, \mathcal{O})$ by

$$\phi_\nu = \sum_{\mu=1}^{N} (-1)^{\mu-1} \sum_{i_1 < \ldots < i_\mu} k_{i_1} \delta k_{i_2} \cdots \delta k_{i_\mu}.$$

Then ϕ_ν satisfies the requirement. Q.E.D.

REMARK. By using $\sup|\operatorname{Re} f_{i_0 \ldots i_\nu}|$ on $W_{i_0 \ldots i_\nu}$ and $\sup|\operatorname{Re} \phi_\nu(f)_{i_0 \ldots i_{\nu-1}}|$ on $W_{i_0 \ldots i_{\nu-1}}$ instead of using $\|f_{i_0 \ldots i_\nu}\|_{W_{i_0 \ldots i_\nu}}$ and $\|\phi_\nu(f)_{i_0 \ldots i_{\nu-1}}\|_{W_{i_0 \ldots i_{\nu-1}}}$, a theorem similar to Theorem 3 can be proved. We need only prove a lemma which corresponds to Lemma 1 but uses sup norms of the real parts instead. To do this, we observe that $f \mapsto \operatorname{Re} f$ defines a continuous $\boldsymbol{R}$-linear *injection* with *closed* image from the Fréchet space E of all holomorphic functions on Q whose constant coefficients in the Laurent series expansions are real to the Fréchet space of all harmonic functions on Q. Hence, for $r < a < b < 1$, there exists a constant C such that, if $f \in E$ and $\sup|\operatorname{Re} f| \leqq K$, then $|f(z)| \leqq CK$ on $a \leqq |z| \leqq b$. The desired lemma follows from an argument analogous to the proof of Lemma 1, but this time we leave a and b fixed instead of letting $a \to r$ and $b \to 1$ and do not restrict $|z|$ to (a, b).

PROOF OF THEOREM 1. By Cartan's Theorem B there is a holomorphic function $\bar{f}$ on U^N such that $Z(\bar{f}) = V$. We can assume $V \cap (Q_{r'})^N = \varnothing$ for some $r' < r$. We are going to prove $(1)_k$ by induction on k.

$(1)_k$ On $U^k \times Q^{N-k}$ (and likewise on products obtained by permuting the N factors) we can construct a bounded holomorphic function $f^{(k)}$ such that $Z(f^{(k)}) = (U^k \times Q^{N-k}) \cap V$ and $(f^{(k)})^{-1}$ is bounded on Q^N.

$Q^N \cap V = \varnothing$ implies that $(U \times Q^{N-1}) \cap V$ is an analytic cover over Q^{N-1} of, say, n sheets. There exists a proper subvariety A in Q^{N-1} and locally defined holomorphic functions $g^{(1)}, \cdots, g^{(n)}$ on Q^{N-1} such that $(U \times (Q^{N-1} - A)) \cap V = \{(z_1, \cdots, z_N) \in U \times (Q^{N-1} - A) \mid z_1 = g^{(i)}(z_2, \cdots, z_N)$ for some $i\}$. The bounded holomorphic extension $f^{(1)}$ on $U \times Q^{N-1}$ of $\prod_{i=1}^n (z_1 - g^{(i)}(z_2, \cdots, z_N))$ satisfies $Z(f^{(1)}) = (U \times Q^{N-1}) \cap V$ and $(f^{(1)})^{-1}$ is bounded on Q^N. $(1)_1$ is proved. Suppose $(1)_k$ is true for $1 \leqq k < m$. Then for $1 \leqq i \leqq m$ we can construct a bounded holomorphic function f_i on $G_i = U^{i-1} \times Q \times U^{m-i} \times Q^{N-m}$ such that $Z(f_i) = G_i \cap V$ and f_i^{-1} is bounded on Q^N. By replacing f_i by the product of f_i with suitable powers of $z_i, z_{m+1}, \cdots, z_N$, we can assume that we can select a regular branch h_i of $\log(\bar{f}/f_i)$ on G_i. Since $h_i - h_j$ = a branch of $\log(f_j/f_i)$ has bounded real part on $G_i \cap G_j$, by the Remark following Theorem 3 we can construct holomorphic functions

$\tilde{h}_i$ on G_i with bounded real parts such that $\tilde{h}_i-\tilde{h}_j=h_i-h_j=$ a branch of $\log(f_j/f_i)$. The holomorphic function $f^{(m)}$ on $U^m\times Q^{N-m}$ which agrees with $f_i\exp(\tilde{h}_i)$ on G_i satisfies $Z(f^{(m)})=(U^m\times Q^{N-m})\cap V$ and is bounded. Moreover, $(f^{(m)})^{-1}$ is bounded on Q^N. $(1)_m$ is proved. The theorem follows from $(1)_N$. Q.E.D.

Proof of Theorem 2. By Theorem 1 we can construct $g\in H^\infty(U^N)$ such that $Z(g)=V$. The construction implies that g^{-1} is bounded on Q^N. Take $f\in H^\infty(V)$. By Cartan's Theorem B f is the restriction to V of a holomorphic function $\tilde{f}$ on U^N. We are going to prove $(2)_k$ by induction on k.

$(2)_k$ On $U^k\times Q^{N-k}$ (and likewise on products obtained by permuting the N factors) we can construct a bounded holomorphic function $f^{(k)}$ which agrees with f on $(U^k\times Q^{N-k})\cap V$.

From the conditions of special Rudin subvarieties we conclude that $(U\times Q^{N-1})\cap V$ is an unbranched analytic cover over Q^{N-1} of, say, n sheets. There are locally defined holomorphic functions $g^{(1)},\cdots,g^{(n)}$ on Q^{N-1} such that $(U\times Q^{N-1})\cap V=\{(z_1,\cdots,z_N)\in U\times Q^{N-1}\mid z_1=g^{(i)}(z_2,\cdots,z_N)$ for some $i\}$. The function $f^{(1)}(z_1,\cdots,z_N)=\sum_{i=1}^N f(g^{(i)}(z_2,\cdots,z_N),z_2,\cdots,z_N)(\prod_{j\neq i,1\leq j\leq n}(z_1-g^{(j)}(z_2,\cdots,z_N)))(\prod_{j\neq i,1\leq j\leq n}(g^{(i)}(z_2,\cdots,z_N)-g^{(j)}(z_2,\cdots,z_N)))^{-1}$ is well defined, agrees with f on $(U\times Q^{N-1})\cap V$, and is bounded. $(2)_1$ is proved. Suppose $(2)_k$ is true for $1\leqq k<m$. We can construct bounded holomorphic functions f_i on $G_i=U^{i-1}\times Q\times U^{m-i}\times Q^{N-m}$, $1\leqq i\leqq m$, such that $f_i=f$ on $G_i\cap V$. Let $h_i=(\tilde{f}-f_i)/g$ on G_i. Since $h_i-h_j=(f_j-f_i)/g$ is bounded on $G_i\cap G_j$ (because g^{-1} is bounded on Q^N), we can construct by Theorem 3 $\tilde{h}_i\in H^\infty(G_i)$ such that $\tilde{h}_i-\tilde{h}_j=h_i-h_j=(f_j-f_i)/g$. The holomorphic function $f^{(m)}$ on $U^m\times Q^{N-m}$ which agrees with $f_i+g\tilde{h}_i$ on G_i is bounded and agrees with f on $(U^m\times Q^{N-m})\cap V$. $(2)_m$ is proved. By $(2)_N$ we can construct $f^{(N)}\in H^\infty(U^N)$ which agrees with f on V. It is clear from the constructions that the map defined by $f\mapsto f^{(N)}$ is a bounded linear map from $H^\infty(V)$ to $H^\infty(U^N)$. Q.E.D.

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