REMARK ON A PAPER OF Y. IKEBE

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In a recent paper, Y. Ikebe [**3**] has proved a theorem on characterization of best approximations by elements of linear subspaces in the space of all continuous real- or complex-valued functions on a compact space. In the present paper we want to observe that this result can be extended to arbitrary real or complex normed linear spaces. We shall use the notations of [**5**].

THEOREM. *Let E be a normed linear space, G a linear subspace of E and $x\in E\setminus\overline{G}$. An element $g_0\in G$ is a best approximation of x (i.e. $\|x-g_0\| =\inf_{g\in G}\|x-g\|$) if and only if* 0 *belongs to the $\sigma(G^*, G)$-closure of the convex hull of the set*

$$(1)\quad A = \{[f(x-g_0)]^- f|_G \mid f\in \mathcal{E}(S_{E^*}),\ |f(x-g_0)| = \|x-g_0\|\},$$

where $\mathcal{E}(S_{E^*})$ denotes the set of all extreme points of the unit cell $S_{E^*} = \{f\in E^* \mid \|f\|\leqq 1\}$, $f|_G$ denotes the restriction of f to the subspace G and where $[\quad]^-$ stands for complex conjugate.

PROOF. Let us denote by $\Omega(A)$ the $\sigma(G^*, G)$-closure of the convex hull of A.

Necessity. Assume that $0\notin\Omega(A)$. Then, as observed also in [**3**], there exists an element $g_1\in G$ such that

$$(2)\qquad \sup_{\phi\in\Omega(A)} \operatorname{Re}\phi(g_1) < 0.$$

Now, if g_0 would be a best approximation of x, then, by [**5**, pp. 57–58, Corollary 1.9], for every $g\in G$ there would exist an $f^g\in\mathcal{E}(S_{E^*})$ such that

$$(3)\qquad \begin{gathered} |f^g(x-g_0)| = \|x-g_0\|, \\ \operatorname{Re}([f^g(x-g_0)]^- f^g(g)) \geqq 0, \end{gathered}$$

whence, for $\phi^g = [f^g(x-g_0)]^- f^g|_G$ one would obtain

$$(4)\qquad \phi^g\in A, \qquad \operatorname{Re}\phi^g(g)\geqq 0,$$

which contradicts (2).

Sufficiency can be proved similarly to [**3**]: assuming that $0\in\Omega(A)$ and using the $\sigma(E^*, E)$-compactness of S_{E^*}, one easily obtains an $f\in E^*$ such that

Received by the editors December 10, 1967.

$$\begin{aligned}\|f\| = 1, \quad f(g) = 0 \quad (g \in G),\\ f(x - g_0) = \|x - g_0\|,\end{aligned} \tag{5}$$

whence [4] g_0 is a best approximation of x. This completes the proof of the theorem.

In the particular case when $E = C(Q)$, the space of all continuous real- or complex-valued functions on a compact space Q, we have $f \in \mathcal{E}(S_{E^*})$ if and only if f is of the form $\alpha_0\epsilon_{q_0}$, i.e.

$$f(y) = \alpha_0 y(q_0) \qquad (y \in C(Q)), \tag{6}$$

where α_0 is a scalar with $|\alpha_0| = 1$ and $q_0 \in Q$ (see e.g. [2, p. 441, Lemma 6], or, for another proof, [5, p. 73]). Therefore

$$\begin{aligned} A &= \{[\alpha(x(q) - g_0(q))]^{-}\alpha\epsilon_q|_G \mid |\alpha| = 1, q \in Q, \\ &\qquad\qquad |x(q) - g_0(q)| = \|x - g_0\|\} \\ &= \{[x(q) - g_0(q)]^{-}\epsilon_q|_G \mid q \in Q, |x(q) - g_0(q)| = \|x - g_0\|\}, \end{aligned} \tag{7}$$

and thus the above theorem yields the result of Y. Ikebe [3].

In the particular case when $\dim G = n < \infty$, the corollary given in [3] with reference to E. W. Cheney [1] can be also extended to arbitrary normed linear spaces as follows:

Corollary. *Let E be a normed linear space, G a linear subspace of E with* $\dim G = n < \infty$, *and $x \in E \setminus G$. An element $g_0 \in G$ is a best approximation of x if and only if* 0 *belongs to the convex hull of the set* (*in the n-space*)

$$\begin{aligned} B = \{[f(x - g_0)]^{-}f(x_1), \cdots, [f(x - g_0)]^{-}f(x_n) \mid f \in \mathcal{E}(S_{E^*}), \\ |f(x - g_0)| = \|x - g_0\|\}, \end{aligned} \tag{8}$$

where $x_1, \cdots, x_n \in G$ is some basis of G.

This corollary can be deduced from the above theorem as in the corresponding particular case in [3], observing that if $\dim G < \infty$, then the convex hull of the set A defined by (1) is $\sigma(G^*, G)$-closed and using the isomorphic mapping $\phi \rightarrow (\phi(x_1), \cdots, \phi(x_n))$ of G^* onto the n-space.

References

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