NOTE ON NONLINEAR CONTRACTION SEMIGROUPS

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1. Let X be a Banach space and let $\{T(\xi); 0 \leqq \xi < \infty\}$ be a family of nonlinear operators from X into itself satisfying the following conditions:

(i) $T(0) = I$ (the identity) and $T(\xi + \eta) = T(\xi)T(\eta)$ for $\xi, \eta \geqq 0$.

(ii) $\|T(\xi)x - T(\xi)y\| \leqq \|x - y\|$ for $x, y \in X$.

(iii) $\lim_{\xi \to 0+} \|T(\xi)x - x\| = 0$ for $x \in X$.

It is clear that for any fixed $x \in X$, $T(\xi)x$ is strongly continuous in $\xi \geqq 0$. Such a family $\{T(\xi); 0 \leqq \xi < \infty\}$ is called a *nonlinear contraction semigroup*. Then we define the *infinitesimal generator* A of $\{T(\xi); 0 \leqq \xi < \infty\}$ by

$$Ax = \lim_{\delta \to 0+} A_\delta x$$

whenever the limit exists, where $A_\delta = \delta^{-1}(T(\delta) - I)$. We denote the domain of A by $D(A)$.

In case of *linear* contraction semigroup, it is well known that

$$T(\xi)x = \lim_{\delta \to 0+} T(\xi; A_\delta)x \quad \text{for } x \in X,\ \xi \geqq 0,$$

and for each fixed $x \in X$, the convergence is uniform with respect to ξ in every compact subset of $[0, \infty)$, where

$$T(\xi; A_\delta) = \exp(\xi A_\delta)$$

(see [1]). In this case it is clear that $\{T(\xi; A_\delta); 0 \leqq \xi < \infty\}$ is a linear contraction semigroup and $T(\xi; A_\delta)$ is continuous in $\xi \geqq 0$ with respect to the uniform operator topology.

In this paper we shall give similar results for the *nonlinear* case. The theorem is as follows.

THEOREM. *Let $\{T(\xi); 0 \leqq \xi < \infty\}$ be a nonlinear contraction semigroup.*

I. *For each $\delta > 0$ there exists a nonlinear contraction semigroup $\{T(\xi; A_\delta); 0 \leqq \xi < \infty\}$ with the infinitesimal generator A_δ $(= \delta^{-1}[T(\delta) - I])$ satisfying the following conditions*:

(a) *For each $x \in X$, $T(\xi; A_\delta)x$ is strongly continuously differentiable in $\xi \geqq 0$ and*

$$(1) \qquad dT(\xi; A_\delta)x/d\xi = A_\delta T(\xi; A_\delta)x \quad \text{for } \xi \geqq 0.$$

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(b) *For each bounded set B in X and $\xi \geqq 0$,*

$$\sup_{x\in B} \|T(\xi + h; A_\delta)x - T(\xi; A_\delta)x\| \to 0 \tag{2}$$

as $h \to 0$.

II. *Suppose that*

(iv) *there exists a set D such that $D \subset D(A)$ and for any $x \in D$, $T(\xi)x \in D(A)$ for a.e. $\xi > 0$.*

Then for each $x \in D^-$ (the closure of D) we have

$$T(\xi)x = \lim_{\delta\to 0+} T(\xi; A_\delta)x \quad \textit{for } \xi \geqq 0, \tag{3}$$

and the convergence is uniform with respect to ξ in every compact subset of $[0, \infty)$.

Hence, if D is dense in X, then (3) *holds good for all $x \in X$.*

The proof is given in §3.

REMARK. The author found in Notices of the American Mathematical Society that J. R. Dorroh [2] has obtained the same result under the assumption

'for each $x \in D$, $T(\xi)x$ is strongly continuously differentiable in $\xi \geqq 0$.'

Y. Kōmura [4] has proved that if X is a Hilbert space, then for each $x \in D(A)$

$$T(\xi)x \in D(A) \quad \text{for a.e. } \xi \geqq 0, \text{ and } T(\xi)x = x + \int_0^\xi AT(\eta)x d\eta. \tag{4}$$

(He has shown that (4) holds good for each $x \in D(A_\Phi)$.) It follows from his argument that (4) is also true for reflexive Banach spaces. Hence, if X is a reflexive Banach space, then our assumption (iv) is always satisfied by taking $D = D(A)$. Thus we have the following

COROLLARY. *If $\{T(\xi); 0 \leqq \xi < \infty\}$ is a nonlinear contraction semi-group defined on a reflexive Banach space X, then for each $x \in (D(A))^-$*

$$T(\xi)x = \lim_{\delta\to 0+} T(\xi; A_\delta)x \quad \textit{for } \xi \geqq 0,$$

and the convergence is uniform with respect to ξ in every compact subset of $[0, \infty)$.

Especially, if $D(A)$ is dense in X, then the above convergence holds good for all $x \in X$.

REMARK. Y. Kōmura [4] proved that $(D(A))^- = X$ if $X = R^n$ (the n-dimensional Euclidean space).

2. The proof of the main part II in our theorem is based on the following Kato's lemmas [**3**].

LEMMA 1. *Let T be an operator with domain $D(T)$ and range $R(T)$ in X. The following conditions are equivalent*:

(D) $\|x-y-\alpha(Tx-Ty)\| \geqq \|x-y\|$ *for every* $x, y \in D(T)$ *and* $\alpha > 0$. (*That is,* $-T$ *is monotonic in the sense of Kato.*)

(D′) *For each* $x, y \in D(T)$, *there is* $f \in F(x-y)$ *such that*

$$\mathrm{Re}\,(Tx-Ty, f) \leqq 0,$$

where F is the duality map from X into X^*.

LEMMA 2. *Let $x(\xi)$ be an X-valued function on an interval of real numbers. Suppose $x(\xi)$ has a weak derivative $x'(\eta) \in X$ at $\xi = \eta$. If $\|x(\xi)\|$ is also differentiable at $\xi = \eta$, then*

$$\|x(\eta)\| \left[d\|x(\xi)\|/d\xi\right]_{\xi=\eta} = \mathrm{Re}(x'(\eta), f)$$

for every $f \in Fx(\eta)$.

Since $T(\delta)$ is a contraction and $A_\delta = \delta^{-1}[T(\delta) - I]$, we have

$$\|A_\delta x - A_\delta y\| \leqq 2\|x-y\|/\delta \quad \text{for } x, y \in X, \tag{5}$$

and

$$\|x - y - \alpha(A_\delta x - A_\delta y)\| \geqq \|x-y\| \tag{6}$$

for $x, y \in X$ and $\alpha > 0$.

Now we shall prove the following

LEMMA 3. *The equation*

$$\begin{aligned} \frac{du(\xi; x)}{d\xi} &= A_\delta u(\xi; x) \quad \textit{for } \xi \geqq 0 \\ u(0; x) &= x \end{aligned} \tag{7}$$

has a unique solution $u(\xi; x) \in C^1([0, \infty); X)$ *for any* $x \in X$, *and*

$$\|u(\xi; x) - u(\xi; y)\| \leqq \|x-y\| \quad \textit{for } x, y \in X \textit{ and } \xi \geqq 0, \tag{8}$$

where $C^1([0, \infty); X)$ denotes the set of all strongly continuously differentiable X-valued functions defined on $[0, \infty)$.

PROOF. Since the map $x \rightarrow A_\delta x$ is Lipschitz continuous, uniformly in x (see (5)), the equation (7) has a unique solution $u(\xi; x) \in C^1([0, \infty); X)$ for any $x \in X$.

We shall prove (8). Fix $x, y \in X$ and put

$$z(\xi) = u(\xi; x) - u(\xi; y).$$

Clearly $z(\xi) \in C^1([0, \infty); X)$ and

$$dz(\xi)/d\xi = A_\delta u(\xi; x) - A_\delta u(\xi; y),$$
$$z(0) = x - y.$$

Since $\|z(\xi)\|$ is absolutely continuous, $\|z(\xi)\|$ is differentiable for a.e. $\xi \geqq 0$. Therefore, by Lemma 2, we get for a.e. $\eta \geqq 0$

$$\begin{aligned} \|z(\eta)\| [d\|z(\eta)\|/d\eta] &= \mathrm{Re}(z'(\eta), f_\eta) \\ &= \mathrm{Re}(A_\delta u(\eta; x) - A_\delta u(\eta; y), f_\eta) \end{aligned}$$

for every $f_\eta \in Fz(\eta) = F(u(\eta; x) - u(\eta; y))$. Therefore it follows from (6) and Lemma 1 that

$$\|z(\eta)\| [d\|z(\eta)\|/d\eta] \leqq 0 \qquad \text{for a.e. } \eta \geqq 0.$$

Since

$$\begin{aligned} \|z(\xi)\|^2 - \|z(0)\|^2 &= \int_0^\xi [d\|z(\eta)\|^2/d\eta] d\eta \\ &= 2\int_0^\xi [\|z(\eta)\| \cdot d\|z(\eta)\|/d\eta] d\eta \leqq 0 \end{aligned}$$

for each $\xi > 0$, we obtain

$$\|u(\xi; x) - u(\xi; y)\| \leqq \|x - y\| \qquad \text{for } \xi > 0. \qquad \text{Q.E.D.}$$

REMARK. It is possible to prove the inequality (8) without using the duality map (that is Lemmas 1 and 2).

3. Proof of the theorem.

PROOF OF I. Let $u(\xi; x)$ be the (unique) solution in Lemma 3, and let

$$T(\xi; A_\delta)x = u(\xi; x) \qquad \text{for } \xi \geqq 0 \quad \text{and} \quad x \in X. \tag{9}$$

It is clear that $\{T(\xi; A_\delta); 0 \leqq \xi < \infty\}$ is a nonlinear contraction semigroup satisfying the condition (a).

We shall show (b).

$$\begin{aligned} \|T(\xi + h; A_\delta)x - T(\xi; A_\delta)x\| &= \left\| \int_\xi^{\xi+h} A_\delta T(\eta; A_\delta) x d\eta \right\| \\ &\leqq \left| \int_\xi^{\xi+h} \|A_\delta T(\eta; A_\delta)x\| d\eta \right| \leqq |h| \, \|A_\delta x\| \end{aligned} \tag{10}$$

because $\|A_\delta T(\eta;\ A_\delta)x\| \leqq \|A_\delta x\|$. If B is a bounded set, then $\sup_{x\in B} \|A_\delta x\|\ (=M_B) < \infty$; hence, by (10), we have

$$\sup_{x\in B} \|T(\xi + h;\ A_\delta)x - T(\xi;\ A_\delta)x\| \leqq M_B |h| \to 0$$

as $h \to 0$.

PROOF OF II. Now assume

(iv) there exists a set D such that $D \subset D(A)$ and for each $x \in D$, $T(\xi)x \in D(A)$ for a.e. $\xi > 0$.

Let $x \in D$.

$$\text{(11)} \quad AT(\xi)x = \lim_{h\to 0+} \frac{T(h) - I}{h} T(\xi)x = \lim_{h\to 0+} \frac{T(\xi + h)x - T(\xi)x}{h}$$

for a.e. $\xi \geqq 0$; and

$$\|AT(\xi)x\| \leqq \|Ax\| \qquad \text{for a.e. } \xi \geqq 0,$$

since $\|h^{-1}[T(\xi+h)x - T(\xi)x]\| \leqq \|h^{-1}[T(h)x - x]\|$. Hence $AT(\xi)x$ is strongly measurable and bounded, and so $AT(\xi)x$ is Bochner integrable on any finite interval. Let $f \in X^*$. It is easy to see that $T(\xi)x$ is strongly absolutely continuous in ξ and a fortiori $(T(\xi)x, f)$ $(=f(T(\xi)x))$ is absolutely continuous. Hence $(T(\xi)x, f)$ is differentiable at a.e. $\xi \geqq 0$ and

$$(T(\xi)x, f) - (x, f) = \int_0^\xi \frac{d}{d\eta} (T(\eta)x, f)d\eta$$

for all $\xi \geqq 0$. Moreover, it follows from (11) that

$$(d/d\eta)(T(\eta)x, f) = (AT(\eta)x, f) \qquad \text{for a.e. } \eta \geqq 0.$$

Thus the above equalities and the Bochner integrability of $AT(\xi)x$ show that

$$\begin{aligned}(T(\xi)x, f) - (x, f) &= \int_0^\xi (AT(\eta)x, f)d\eta \\ &= \left(\int_0^\xi AT(\eta)xd\eta, f\right) \qquad \text{for all } \xi \geqq 0.\end{aligned}$$

Hence we obtain

$$\text{(12)} \qquad T(\xi)x - x = \int_0^\xi AT(\eta)xd\eta \qquad \text{for all } \xi \geqq 0,$$

and then $(d/d\xi)T(\xi)x = AT(\xi)x$ for a.e. $\xi \geqq 0$.

If we put

$$z_\delta(\xi) = T(\xi; A_\delta)x - T(\xi)x \qquad \left(= \int_0^\xi [A_\delta T(\eta; A_\delta)x - AT(\eta)x]d\eta\right)$$

for $\xi \geqq 0$, then $z_\delta(\xi)$ has strong derivative

$$z_\delta'(\xi) = A_\delta T(\xi; A_\delta)x - AT(\xi)x \qquad \text{for a.e. } \xi \geqq 0,$$

and $\|z_\delta(\xi)\|$ is also differentiable for a.e. $\xi \geqq 0$ since $\|z_\delta(\xi)\|$ is absolutely continuous with respect to ξ. Using Lemma 2, for a.e. $\xi \geqq 0$

$$(13) \qquad \begin{aligned} \|z_\delta(\xi)\| [d\|z_\delta(\xi)\|/d\xi] &= \mathrm{Re}(z_\delta'(\xi), f_\xi) \\ &= \mathrm{Re}(A_\delta T(\xi; A_\delta)x - AT(\xi)x, f_\xi) \end{aligned}$$

for every $f_\xi \in Fz_\delta(\xi) = F(T(\xi; A_\delta)x - T(\xi)x)$. Moreover, by (6) and Lemma 1, for each $\xi \geqq 0$ there is $f_\xi^0 \in Fz_\delta(\xi) = F(T(\xi; A_\delta)x - T(\xi)x)$ such that

$$\mathrm{Re}(A_\delta T(\xi; A_\delta)x - A_\delta T(\xi)x, f_\xi^0) \leqq 0.$$

Combining this and (13), we obtain

$$\|z_\delta(\xi)\| [d\|z_\delta(\xi)\|/d\xi] \leqq \mathrm{Re}(A_\delta T(\xi)x - AT(\xi)x, f_\xi^0)$$

for a.e. $\xi \geqq 0$; and hence

$$(14) \qquad \begin{aligned} \|z_\delta(\xi)\| [d\|z_\delta(\xi)\|/d\xi] &\leqq \|A_\delta T(\xi)x - AT(\xi)x\| \|f_\xi^0\| \\ &= \|A_\delta T(\xi)x - AT(\xi)x\| \|T(\xi; A_\delta)x - T(\xi)x\| \end{aligned}$$

for a.e. $\xi \geqq 0$. Thus for each $\xi \geqq 0$

$$\begin{aligned} \|z_\delta(\xi)\|^2 &= \int_0^\xi [d\|z_\delta(\eta)\|^2/d\eta]\, d\eta = 2\int_0^\xi \|z_\delta(\eta)\| [d\|z_\delta(\eta)\|/d\eta] d\eta \\ &\leqq 2\int_0^\xi \|A_\delta T(\eta)x - AT(\eta)x\| \, \|T(\eta; A_\delta)x - T(\eta)x\| d\eta; \end{aligned}$$

so that for any $\beta > 0$

$$(15) \qquad \begin{aligned} &\sup_{0 \leqq \xi \leqq \beta} \|T(\xi; A_\delta)x - T(\xi)x\|^2 \\ &\qquad \leqq 2\int_0^\beta \|A_\delta T(\eta)x - AT(\eta)x\| \|T(\eta; A_\delta)x - T(\eta)x\| d\eta \end{aligned}$$

We have

$$\lim_{\delta\to 0+} \|A_\delta T(\eta)x - AT(\eta)x\| = 0 \qquad \text{for a.e. } \eta \geqq 0,$$

$$\|AT(\eta)x\| \leqq \|Ax\| \qquad \text{for a.e. } \eta \geqq 0,$$

$$\|A_\delta T(\eta)x\| \leqq \|A_\delta x\| \quad (=O(1) \text{ as } \delta \to 0+) \qquad \text{for all } \eta \geqq 0,$$

and

$$\|T(\eta;\, A_\delta)x\| \leqq \|x\| + \int_0^\eta \|A_\delta T(\tau;\, A_\delta)x\| d\tau$$

$$\leqq \|x\| + \|A_\delta x\|\beta \qquad \text{for } \eta \in [0, \beta].$$

These show that the integrand of (15) converges boundedly to zero as $\delta \to 0+$. Thus it follows from the convergence theorem that for any $\beta > 0$

$$\sup_{0\leqq\xi\leqq\beta} \|T(\xi;\, A_\delta)x - T(\xi)x\| \to 0 \tag{16}$$

as $\delta \to 0+$.

Finally, since $T(\xi;\, A_\delta)$ and $T(\xi)$ are contraction operators, it is clear that (16) holds good for each $x \in D^-$. Q.E.D.

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ADDED IN PROOF. Recently the author proved the following: "If C is a contraction operator (not necessarily linear) from X into itself, then $\|T(m;\, C-I)x - C^m x\| \leqq m^{1/2}\|(C-I)x\|$ for every $x \in X$ and positive integer m, where $\{T(\xi;\, C-I);\, 0 \leqq \xi < \infty\}$ is a nonlinear contraction semigroup generated by $C-I$."

In the case of linear operators, this has been obtained by P. R. Chernoff (*Note on product formulas for operator semigroups*, J. Functional Analysis 2 (1968), 238–242).

Then it follows from the above result that, without the assumption (iv), the convergence (3) in the theorem holds for each $x \in D(A)^-$ (i.e., the corollary remains true for *general* Banach space X).

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