A SEMIGROUP ANALOGUE OF FOGUEL'S COUNTEREXAMPLE

EDWARD W. PACKEL[1]

1. Introduction. In [1] Foguel provides a counterexample to Nagy's question of whether every power bounded operator on a Hilbert space is similar to a contraction. Generally questions about powers of operators have exponential analogues which can be phrased as questions about (C_0) semigroups. The purpose of this note is to provide a counterexample to the semigroup analogue of Nagy's question—that is, to construct a uniformly bounded (C_0) semigroup on a Hilbert Space H whose generator is not similar to a dissipative operator. (An operator L is dissipative on H if $\langle Ly, y\rangle + \langle y, Ly\rangle \leqq 0$ for all $y \in \text{Domain } (L)$.) The method used depends strongly on Foguel's ideas and utilizes the viewpoint presented in Halmos' note [2] on Foguel's counterexample.

2. A necessary condition for similarity to a dissipative operator. Let $S(t)$ denote a (C_0) semigroup and L its infinitesimal generator on H. We will need the characterization of dissipative semigroup generators presented by Phillips [3, p. 203] which says that L is dissipative if and only if $\|S(t)\| \leqq 1$ for all $t \geqq 0$.

Paralleling Foguel we define

$$W(L) = \{x \in H\text{: weak limit } (t \to \infty)\ S(t)x = 0\}.$$

As a necessary condition for similarity to a dissipative operator we prove:

LEMMA 1. *If L is similar to a dissipative operator, then* $W(L) \cap [W(L^*)]^{\perp} = \{0\}$.

PROOF. The argument is broken down into three parts in the manner of Halmos.

Part I. If A generates a group of unitary operators $U(t)$, $-\infty < t < \infty$, then $W(A) = W(A^*)$ where A^* generates the group $U^*(t)$. To see this, represent $U(t)$ as multiplication by u^t on some $L_2(\mu)$ isometrically isomorphic to H, where u is a measurable function of constant modulus one. Showing $W(A) \subseteq W(A^*)$ means showing that

Received by the editors March 18, 1968.

[1] This work is a part of the author's doctoral dissertation submitted to the Massachusetts Institute of Technology in August 1967. The author wishes to thank Professor Gilbert Strang for his help and encouragement.

$\int u^t f\bar{g}d\mu \to 0$ for all $g \in L_2(\mu)$ implies $\int \bar{u}^t f\bar{h}d\mu \to 0$ for all $h \in L_2(\mu)$. This follows by setting, for each h, $g = (\operatorname{sgn} f)^2\bar{h}$ and taking complex conjugates. By symmetry $W(A^*) \subseteq W(A)$, so equality holds.

Part II. If D is dissipative and hence generates a contraction semigroup $C(t)$, then $W(D) = W(D^*)$. For this argument, the theory of strong unitary dilations [**4**, p. 15] yields a strongly continuous group $U(t)$ of unitary operators on a Hilbert space $H' \supset H$ such that $PU(t)|_H = C(t)$ for all $t \geqq 0$ where P is the orthogonal projection of H' onto H. Furthermore H' is minimal in the sense that it is spanned by the elements $\{U(t)x\}$, $x \in H$ and $-\infty < t < \infty$. Define $H_x = \{y \in H' : \langle U(t)x, y\rangle \to 0 \text{ as } t \to \infty\}$. For $x \in W(D)$, $\langle C(t)x, y\rangle = \langle PU(t)x, y\rangle = \langle U(t)x, Py\rangle = \langle U(t)x, y\rangle \to 0$ as $t \to \infty$ for all $y \in H$. Hence $H \subseteq H_x$ whenever $x \in W(D)$. Furthermore, H_x is a linear manifold invariant under $U(t)$ and $U^*(t)$, $-\infty < t < \infty$, and since H' is minimal as stated above, it follows that $x \in W(D)$ implies $H_x = H'$. Call A the generator of $U(t)$ and note that we then have $W(D) \subseteq W(A) \cap H$ where $W(A)$ is defined on H'. Conversely, $x \in W(A) \cap H$ implies $(U(t)x, y) \to 0$ as $t \to \infty$ for all $y \in H'$. Thus for $y \in H$, $(C(t)x, y) = (PU(t)x, y) = (u(t)x, y) \to 0$ as $t \to \infty$, so that $x \in W(D)$. This shows that $W(A) \cap H = W(D)$ and using the same fact for A^* and D^* and Part I, $W(D) = W(D^*)$ follows.

Part III. The completion of the lemma now follows by noting that if $L = SDS^{-1}$, D dissipative, then $W(L) = S(W(D))$ and $[W(L^*)]^\perp = S([W(D^*)]^\perp)$. Since, by Part II, $W(D) = W(D^*)$ so that $W(D) \cap [W(D^*)]^\perp = \{0\}$, we obtain $W(L) \cap [W(L^*)]^\perp = \{0\}$.

3. **Construction of the counterexample.** Let H be the Hilbert space $L_2([0, \infty))$ and let $V(t)$ be right translation by t on H, so for $t \geqq 0$,

$$\begin{aligned} V(t)f(x) &= f(x - t) \qquad \text{when } x \geqq t, \\ &= 0 \qquad \text{otherwise,} \end{aligned}$$

and

$$V^*(t)f(x) = f(x + t).$$

It is easy to check that $V(t)$ and $V^*(t)$ are (C_0) semigroups of operators whose norms are identically one.

For each $t > 0$, define the integer $k_0 = k_0(t)$ by requiring $4^{k_0} < t \leqq 4^{k_0+1}$. For each integer $k > k_0(t)$ define the interval $I_k(t) = (4^k - t, 4^k]$ and let $I_{k_0}(t) = [0, 2 \cdot 4^{k_0} - t]$. Define the operator $P(t)$ on H for each $t \geqq 0$ by $P(0) = 0$ and

$$\begin{aligned} P(t)f(x) &= f(2 \cdot 4^k - t - x) \qquad \text{when } x \in I_k(t), \quad k \geqq k_0(t), \\ &= 0 \qquad \text{otherwise.} \end{aligned}$$

By inspection $P(t)$ is well defined and $\|P(t)\| \leqq 1$ for all $t \geqq 0$.

Let $H_2 = H \oplus H$ with the usual direct sum inner product: $\langle (f, g), (h, k) \rangle = \langle f, h \rangle + \langle g, k \rangle$. Define the operators $S(t)$, $t \geqq 0$, on H_2 by

$$S(t) = \begin{pmatrix} V^*(t) & P(t) \\ 0 & V(t) \end{pmatrix},$$

i.e., $S(t)(f, g) = (V^*(t)f + P(t)g,\ V(t)g)$.

Since $\|V(t)\| = \|V^*(t)\| = 1 \geqq \|P(t)\|$ for all $t \geqq 0$ it follows that $\|S(t)\| \leqq 2$ for all $t \geqq 0$. We now establish that $S(0) = I$ and that $S(t)$ has the semigroup property $S(u+t) = S(u)S(t)$ for all u, $t > 0$. Since $P(0) = 0$, we have $S(0) = I$. The semigroup property translates into the following equality:

$$\text{(I)} \qquad V^*(u)P(t)g(x) + P(u)V(t)g(x) = P(u+t)g(x)$$

for all g in H and u, $t \geqq 0$, where the fact that $V(t)$ and $V^*(t)$ are semigroups has been used. Rewriting (I) with the definitions of $V(t)$, $V^*(t)$, and $P(t)$, we must prove that $h_1(x) + h_2(x) = h_3(x)$ where

$$\begin{aligned}
h_1(x) &= g(2 \cdot 4^k - t - x - u) && \text{when } x + u \in I_k(t), \quad k \geqq k_0(t), \\
&= 0 \quad \text{otherwise;} \\
h_2(x) &= g(2 \cdot 4^k - t - x - u) && \text{when } x \in I_k(u), \quad x \leqq 2 \cdot 4^k - t - u, \\
& && \qquad\qquad k \geqq k_0(u), \\
&= 0 \quad \text{otherwise;} \\
h_3(x) &= g(2 \cdot 4^k - t - x - u) && \text{when } x \in I_k(u+t), \quad k \geqq k_0(u+t), \\
&= 0 \quad \text{otherwise.}
\end{aligned}$$

Letting g, u, t, and x be fixed but arbitrary, and noting that the arguments of h_1, h_2, and h_3 are equal, we need only verify that the conditions imposed on x for these functions enable us to obtain the desired equality. Since the computations are not entirely obvious we include some of them, splitting the proof into three parts.

(1) If $h_1(x) \neq 0$, then $h_1(x) = h_3(x)$. Indeed, if $h_1(x) \neq 0$, then $x + u \in I_k(t)$, $k \geqq k_0(t)$. First suppose $k > k_0(t)$ so that $t \leqq 4^k$ and $x \in (4^k - t - u,\ 4^k - u]$ (and $x \geqq 0$). If $4^k - t - u \geqq 0$, then $u + t \leqq 4^k$ and $k > k_0(u+t)$ so that $x \in I_k(u+t)$ and $h_3(x) = h_1(x)$. If $4^k - t - u < 0$, then $4^k < u + t \leqq 4^{k+1}$ (since $u \leqq 4^k$ and $t \leqq 4^k$) and $k_0(u+t) = k$. In this case, $4^k - t \geqq 0$ implies $2 \cdot 4^k - t - u \geqq 4^k - u$ so that $x \in I_{k_0}(u+t)$ and again $h_1(x) = h_3(x)$. Finally, suppose $k = k_0(t)$. Then $x \in [0,\ 4^k - u - t]$ implies $k = k_0(u+t)$ and $x \in I_{k_0}(u+t)$, so $h_1(x) = h_3(x)$. Similar arguments show that if $h_2(x) \neq 0$, then $h_2(x) = h_3(x)$.

(2) Either $h_1(x)$ or $h_2(x)$ must be zero (for each fixed x, t, and u). For if both were nonzero we would have $x + u \in I_k(t)$ for some $k \geqq k_0(t)$

and $x \in I_j(u)$ for some $j \geqq k_0(u)$. If $k > k_0(t)$ and $j > k_0(u)$ then $(4^k - u - t,\ 4^k - u]$ and $(4^j - u,\ 4^j]$ clearly have empty intersection, yielding a contradiction. Likewise, if $k > k_0(t)$ and $j = j_0(u)$, we have $4^j < u \leqq 4^{j+1}$ and $u + t \leqq 2 \cdot 4^j$. Hence $t < 4^j$, from which it follows that $k < j$ and $2 \cdot 4^j > 4^k$ and thus $2 \cdot 4^j - u - t > 4^k - u - t$. Once again $x + u \in I_k(t)$ and $x \in I_j(u)$ are incompatible. The remaining two cases in which $k = k_0(t)$, $j > k_0(u)$ and $k = k_0(t)$, $j = k_0(u)$ follow similarly and we conclude that either $h_1(x)$ or $h_2(x)$ must be zero.

(3) If $h_3(x) \neq 0$ then either $h_1(x)$ or $h_2(x)$ is nonzero. Here we have $x \in I_k(u+t)$, $k \geqq k_0(u+t)$. If $k > k_0(u+t)$, then $k > k_0(u)$ and $k > k_0(t)$ so that $I_k(u+t) = I_k(u) \cup [x : x + u \in I_k(t)]$. Thus one of $h_1(x)$ and $h_2(x)$ must be equal to $h_3(x)$ and hence nonzero. Finally if $k = k_0(u+t)$, then $x \in I_{k_0}(u+t)$. If either $k_0(t) = k_0(u+t)$ or $k_0(u) = k_0(u+t)$ then $x \in I_{k_0}(t)$ or $x \in I_{k_0}(u)$ and $h_1(x)$ or $h_2(x)$ is nonzero. If both $k_0(t) < k_0(u+t)$ and $k_0(u) < k_0(u+t)$, then $I_k(u+t) \subseteq I_k(u) \rightarrow [x : x + u \in I_k(t)]$ and again one of the $h_1(x)$ and $h_2(x)$ must be nonzero.

Taking (1), (2), and (3) together we obtain $h_1(x) + h_2(x) = h_3(x)$ so that $S(t)$ is a semigroup.

To show that $S(t)$ is a (C_0) semigroup we must show that for each (f, g) in H_2, $S(t)(f, g)$ is continuous in t on $[0, \infty)$. By the semigroup property this reduces to showing that $\|S(t)(f, g) - (f, g)\| \rightarrow 0$ as $t \rightarrow 0$. To this end we have

$$\begin{aligned}\|S(t)(f, g) - (f, g)\|^2 &= \|(V^*(t)f + P(t)g - f,\ V(t)g - g)\|^2 \\ &= \|V^*(t)f + P(t)g - f\|^2 + \|V(t)g - g\|^2 \\ &\leqq 2\|V^*(t)f - f\|^2 + 2\|P(t)g\|^2 + \|V(t)g - g\|^2.\end{aligned}$$

The first and third terms of the last line get small as $t \rightarrow 0$ by the continuity of translation on $L_2([0, \infty))$ (or equivalently since $V(t)$ and $V^*(t)$ are (C_0) semigroups). Also $\|P(t)g\| \rightarrow 0$ because $\|P(t)g\| = \|X(t)g\|$ where $X(t)$ is multiplication by the characteristic function of $\bigcup_{k=k_0(t)}^{\infty} I_k(t)$, and $\|X(t)g\| \rightarrow 0$ by the Lebesgue dominated convergence theorem.

Thus $S(t)$ is a uniformly bounded ($\|S(t)\| \leqq 2$ for all $t \geqq 0$) (C_0) semigroup and by the general theory of (C_0) semigroups, $S(t)$ has an infinitesimal generator L defined for $y \in H_2$ by

$$Ly = \lim_{t \downarrow 0} \frac{S(t)y - y}{t} \qquad \text{whenever this limit exists.}$$

THEOREM. *There exists a uniformly bounded* (C_0) *semigroup* $S(t)$ *whose generator* L *is not similar to a dissipative operator.*

PROOF. In view of Lemma 1 it suffices to show that

$$W(L) \cap [W(L^*)]^{\perp} = \{0\}.$$

Setting g = characteristic function of $(0, 1)$ and defining $R = \{2 \cdot 4^k - 1 : k = 0, 1, 2, \cdots\}$, the definition of $P(t)$ shows that $P(r)g(x) = g(x)$ for all $r \in R$. Indeed $I_{k_0}(r) = [0, 1]$ for such r. Now using the fact that $\langle V(r)g, h\rangle \to 0$ as $r \to \infty$ for all $h \in H$,

$$\begin{aligned} &\text{weak limit } (r \to \infty, r \in R) S(r)(0, g) \\ &\qquad = \text{weak limit } (r \to \infty, r \in R)(g, V(r)g) = (g, 0). \end{aligned}$$

Also for $(h, k) \in W(L^*)$,

$$\begin{aligned} \langle (g, 0), (h, k)\rangle &= \text{limit } (r \to \infty, r \in R)\langle S(r)(0, g), (h, k)\rangle \\ &= \text{limit } (r \to \infty, r \in R)\langle (0, g), S^*(r)(h, k)\rangle \\ &= 0 \end{aligned}$$

by the definition of $(h, k) \in W(L^*)$. Thus $(g, 0) \in [W(L^*)]^{\perp}$ and clearly $(g, 0) \in W(L)$, so $(g, 0) \in W(L) \cap [W(L^*)]^{\perp}$ and L is not similar to a dissipative operator.

An investigation into the nature of the generator L of $S(t)$ yields the following interesting result. The domain of L consists of all $(f, g) \in H_2$ for which

(1) g is (modulo null functions) absolutely continuous on H, $\lim_{x \downarrow 0} g(x) = 0$, and $dg/dx \in H$.

(2) f is absolutely continuous everywhere except possibly at the points 4^k, k integral.

(3) Letting $h(x) = f(x) + \sum_{4k<x} g(4^k)$, then $dh/dx \in H$, so that the jumps of f at 4^k are the negatives of the values $g(4^k)$.

For $(f, g) \in$ Domain (L), $L(f, g) = (dh/dx, -dg/dx)$.

In conclusion, the counterexample presented is a (C_0) semigroup with unbounded generator. This leaves open the question of whether or not the (bounded) generator of a uniformly bounded, uniformly continuous semigroup must be similar to a dissipative operator.

References

1. S. R. Foguel, *A counterexample to a problem of Sz.-Nagy*, Proc. Amer. Math. Soc. 15 (1964), 788–790.

2. P. R. Halmos, *On Foguel's answer to Nagy's question*, Proc. Amer. Math. Soc. 15 (1964), 791–793.

3. R. S. Phillips, *Dissipative operators and hyperbolic systems of partial differential equations*, Trans. Amer. Math. Soc. 90 (1959), 193–254.

4. F. Riesz and B. Sz.-Nagy, *On extensions of linear transformations on a Hilbert space which extend beyond that space*, Appendix to *Functional Analysis*, Ungar, New York, 1960.

Reed College