INTEGRABILITY OF ALMOST KAEHLER MANIFOLDS

S. I. GOLDBERG[1]

1. **Introduction.** Apparently the only known class of examples of almost Kaehler manifolds which are not Kaehlerian are the tangent bundles of nonflat Riemannian manifolds [2], [3], flatness being the integrability condition of the almost complex structure. These spaces are not compact; however, for those which are, it is a strong conjecture that if the almost Kaehler metric is an Einstein metric, the manifold is Kaehlerian. Denote by Ω the fundamental 2-form of the almost Kaehler manifold (M, J, g) with metric g and almost complex structure J. Then, by definition, $d\Omega=0$ from which it easily follows that $\delta\Omega=0$ where d and δ are the differential and codifferential operators, respectively. Hence, Ω is harmonic, that is, $\Delta\Omega$ vanishes, where $\Delta=d\delta+\delta d$ is the Laplace-Beltrami operator. Moreover, Ω has constant length; indeed, $|\Omega|^2=\langle\Omega, \Omega\rangle=2n$ where $\langle\ ,\ \rangle$ denotes the local scalar product induced by g and $n=\dim_C M$. In the sequel, a p-form will be called harmonic if it is a zero of the operators d and δ. For compact manifolds, the two definitions of harmonicity are equivalent. If M is Kaehlerian, its fundamental form has vanishing covariant derivative with respect to the Kaehler metric g, so the curvature transformation of g commutes with J. It is the main purpose of this note to prove the converse:

THEOREM 1. *If the curvature transformation of the metric g of the almost Kaehler manifold (M, J, g) commutes with J, then M is a Kaehler manifold.*

COROLLARY 1.1. *The curvature transformation of the almost Kaehler structure of the tangent bundle $T(M)$ of a Riemannian manifold M commutes with the almost complex structure of $T(M)$ if and only if M is locally flat.*

If $\dim_C M>1$ it is easily seen that a Kaehler manifold of constant curvature is locally flat in the given metric. For almost Kaehler manifolds we have

COROLLARY 1.2. *An almost Kaehler manifold of constant curvature is a Kaehler manifold if and only if it is locally flat.*

Observe that in dimension 1 an almost complex structure is com-

Received by the editors March 7, 1968.

[1] Research supported by NSF Grant GP-5477.

plex, so an almost Kaehler manifold of dimension 1 is a Kaehler manifold.

Since a harmonic form on a compact symmetric space has vanishing covariant derivative with respect to the connection of the invariant metric, the fundamental 2-form of a compact symmetric almost Kaehler manifold has vanishing covariant derivative. This is also a consequence of Theorem 1 because of the special nature of the curvature transformation of a symmetric space.

COROLLARY 1.3. *A compact symmetric almost Kaehler manifold is a Kaehler manifold.*

This supports the above conjecture since an irreducible symmetric space is an Einstein space.

2. **Notation and formulae.** An almost Kaehler manifold will be considered as a Riemannian manifold with metric g admitting a skew-symmetric linear transformation field J (the almost complex structure tensor) such that $J^2=-I$; moreover, its fundamental 2-form Ω defined by $\Omega(X, Y)=g(X, JY)$ is closed. The relationship between the curvature transformation $R(X, Y)$ (X, $Y\in M_m$—the tangent space at $m\in M$) and the metric is given by

$$R(X, Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$$

where ∇_X denotes the operation of covariant differentiation in the direction of X and

$$\begin{aligned}2g(X, \nabla_Z Y) = {} & Zg(X, Y) - Xg(Y, Z) + Yg(X, Z) \\ & + g(Z, [X, Y]) - g(X, [Y, Z]) + g(Y, [X, Z]).\end{aligned}$$

Let (M, J, g) be a Kaehler manifold. Then, for any X, $Y\in M_m$,

(a) $R(JX, JY)=R(X, Y)$,

(b) $K(JX, JY)=K(X, Y)$,

where $K(X, Y)$ is the sectional curvature of the plane determined by the vectors X and Y, and when X, Y, JX, JY are orthonormal vectors

(c) $g(R(X, JX)Y, JY)=K(X, Y)+K(X, JY)$. In terms of a basis $\{X_\alpha\}_{\alpha=1,\cdots,d}$ of M_m we set

$$R_{\alpha\beta\gamma\delta} = g(R(X_\alpha, X_\beta)X_\gamma, X_\delta),$$

$$R_{\alpha\beta} = \operatorname{trace}(X_\gamma \to R(X_\alpha, X_\gamma)X_\beta),$$

$$\xi_{\alpha_1\cdots\alpha_p} = \xi(X_{\alpha_1}, \cdots, X_{\alpha_p}),$$

$$\nabla_\beta\xi_{\alpha_1\cdots\alpha_p} = (\nabla_{X_\beta}\xi)(X_{\alpha_1}, \cdots, X_{\alpha_p}).$$

3. A variation of the Bochner-Lichnerowicz technique. Our method is based on the following observation the proof of which is an adaptation of that of Theorem 3.2 in [4] where compactness is required by virtue of a certain maximum principle.

PROPOSITION 2. *A harmonic p-form ξ of constant length on a Riemannian manifold M has vanishing covariant derivative if and only if the quadratic form*

$$F(\xi) = R_{\alpha\beta}\xi^{\alpha\alpha_2\cdots\alpha_p}\xi^{\beta}_{\alpha_2\cdots\alpha_p} - \frac{p-1}{2}R_{\alpha\beta\gamma\delta}\xi^{\alpha\beta\alpha_3\cdots\alpha_p}\xi^{\gamma\delta}_{\alpha_3\cdots\alpha_p}$$

is nonnegative on M.

In fact, for the length function $|\xi|$, we have, since ξ is harmonic

$$p!\cdot\Delta|\xi|^2 = pF(\xi) + g^{\beta\gamma}\nabla_\gamma\xi^{\alpha_1\cdots\alpha_p}\nabla_\beta\xi_{\alpha_1\cdots\alpha_p},$$

so since $|\xi|$ has constant length, the right-hand side vanishes.

A 2-form ξ on an almost complex manifold with almost complex structure J is said to be of *bidegree* (1, 1) if $\xi(X, JY)+\xi(JX, Y)=0$.

Let M be an almost Kaehler manifold. Then, since an orthonormal basis $\{X_i, JX_i\}$ may be chosen at each $m\in M$ such that the only nonvanishing components of a real 2-form of bidegree (1, 1) are of the form $\xi_{ii^*}=\xi(X_i, JX_i)$ (see [1]), we obtain

COROLLARY 2.1. *Let ξ be a harmonic form of bidegree* (1, 1) *on the almost Kaehler manifold M. Then, the quadratic form $F(\xi)$ on M may be expressed in the normal form*

$$2F(\xi) = \sum_i \sum_{j\neq i,i^*} (K_{ij} + K_{ij^*}+K_{i^*j}+K_{i^*j^*})(\xi_{ii^*})^2 - 8\sum_{i<j} R_{ii^*jj^*}\xi_{ii^*}\xi_{jj^*}.$$

REMARK. Observe that $F(\Omega)$ vanishes if M is Kaehler.

COROLLARY 2.2. *If M is Kaehler and curvature is nonnegative the covariant derivative of a harmonic 2-form of bidegree* (1, 1) *and constant length is zero.*

This is an immediate consequence of the identity (c) in §2 (see formula (4.1)).

The Ricci 2-form Ψ, defined by $\Psi(X, Y)=s(X, JY)$ where s is the Ricci tensor, is easily seen to be closed and of bidegree (1, 1). If the manifold is homogeneous Kaehlerian, the scalar curvature is constant, so Ψ is also coclosed. Thus if sectional curvature is nonnegative $\nabla\Psi$ vanishes.

COROLLARY 2.3. *A homogeneous Kaehler manifold with nonnegative curvature with respect to the invariant Kaehlerian metric is either an Einstein space or locally a product of Einstein spaces.*

Since a harmonic form on a compact Riemannian manifold is invariant by the largest connected Lie group of isometries, a harmonic form on a compact homogeneous Riemannian manifold has constant length.

COROLLARY 2.4. *The covariant derivative of a harmonic 2-form of bidegree* (1, 1) *on a compact homogeneous Kaehler manifold with nonnegative curvature vanishes.*

4. **Proof of Theorem 1.** Since J is an isometry, $g(R(X, Y)JZ, JW) = g(JR(X, Y)Z, JW) = g(R(X, Y)Z, W)$, so $R(JZ, JW) = R(Z, W)$. Replacing Y by JY and using the skew-symmetry of $R(X, Y)$ we get $R(X, JY) = R(Y, JX)$. For sectional curvature we have the corresponding relation $K(X, JY) = K(Y, JX)$ and when X, Y, JX, JY are orthonormal, the relation (c) is also seen to hold. Applying Corollary 2.1 we obtain, since $R_{ii^*jj^*} = K_{ij} + K_{ij^*}$,

$$F(\xi) = \sum_{i<j} (K_{ij} + K_{ij^*})(\xi_{ii^*} - \xi_{jj^*})^2. \tag{4.1}$$

Upon setting $\xi = \Omega$ and observing that $\Omega_{ii^*} = \Omega(X_i, JX_i) = g(X_i, X_i) = 1$, we see that $F(\xi) = 0$. Consequently, by Proposition 2, $\nabla\Omega$ vanishes, so M is complex, that is M is Kaehlerian.

Let Φ denote the "Chern 2-form" $\frac{1}{2}\iota(\Omega)R$ where R is the curvature tensor and ι the interior product operator. Then, we have

COROLLARY 1.4. *Let M be an almost Kaehler manifold. Then*

(a) *if $\Phi = \Psi$, M is Kaehlerian;*

(b) *if the 2-form $\Psi - \Phi$ is effective, that is, if $r = \iota(\Omega)\Phi$ where r is the scalar curvature, then M is Kaehlerian.*

REMARK. Observe that in a coordinate neighborhood with the coordinate vectors X, Y, Z, W

$$\begin{aligned} ([\nabla_X, \nabla_Y]\Omega)(Z, W) &= -\Omega([\nabla_X, \nabla_Y]Z, W) - \Omega(Z, [\nabla_X, \nabla_Y]W) \\ &= g(R(X, Y)JZ, W) + g(R(X, Y)Z, JW), \end{aligned}$$

so that an equivalent formulation of the integrability condition in Theorem 1 is given by $R(X, Y)\Omega = 0$ where the curvature transformation applied to forms gets its meaning from its definition in terms of covariant derivatives.

References

1. R. L. Bishop and S. I. Goldberg, *On the second cohomology group of a Kaehler manifold of positive curvature*, Proc. Amer. Math. Soc. **16** (1965), 119–122.

2. S. Sasaki, *On the differential geometry of tangent bundles of Riemannian manifolds*. II, Tôhoku Math. J. **14** (1962), 146–155.

3. S. Tachibana and M. Okumura, *On the almost complex structure of tangent bundles of Riemannian spaces*, Tôhoku Math. J. **14** (1962), 156–161.

4. K. Yano and S. Bochner, *Curvature and Betti numbers*, Ann. of Math. Studies, No. 32, Princeton Univ. Press, Princeton, N. J., 1953.

University of Illinois, Urbana