A NOTE ON CLOSED MAPS AND METRIZABILITY

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1. Introduction. Investigating conditions under which quotients of metric spaces are metrizable, Stone [3] and Morita and Hanai [2] independently obtained the following result.

Theorem A. If a topological space $Y$ is the closed, continuous image of a metric space, and if $Y$ satisfies the first axiom of countability, then $Y$ is metrizable.

The hypothesis of first-countability cannot be dropped. For example, if $Y$ is the quotient space obtained from the real line $\mathbb{R}$ by identifying the integers to a point, then the natural projection

$$\pi : \mathbb{R} \to Y$$

is closed and continuous, but $Y$ is not metrizable. Notice that, although $\pi$ is closed, the product map

$$\pi \times \pi : R \times R \to Y \times Y$$

is not closed. (For example, the set $\{(n, 1/n) | n = 2, 3, 4, \ldots \}$ is closed in $R \times R$ but its image in $Y \times Y$ is not closed.) In fact, by Theorem B below, $Y \times Y$ is not the closed, continuous image of any metric space.

Theorem B. If $X$ and $Y$ are nondiscrete topological spaces and if $X \times Y$ is the closed, continuous image of a metric space, then $X \times Y$ is metrizable.

We shall prove Theorem B, using Theorem A.

2. Semicanonical covers. Let $\mathcal{U}$ be a collection of subsets of a set $S$. For each $W \subseteq S$, we define the star of $W$ with respect to $\mathcal{U}$ by

$$\text{st}(W, \mathcal{U}) = \bigcup \{ V \in \mathcal{U} : W \cap V \neq \emptyset \}. $$

A pair $(Y, B)$ is a topological space $Y$ together with a closed subset $B$. If $Y$ is metrizable, then $(Y, B)$ is called a metric pair.

Let $(Y, B)$ be a pair. As in [1], we call a collection $\mathcal{U} = \{ V_\alpha \}$ of open subsets of $Y$ a semicanonical cover for $(Y, B)$ if

1. $\bigcup_\alpha V_\alpha = Y - B$, and
2. for each $b \in B$ and each neighborhood $^1 U$ of $b$ in $Y$ there exists
a neighborhood $W$ of $b$ in $Y$ such that $\text{st}(W, \mathcal{U}) \subseteq U$.

If a semicanonical cover exists for $(Y, B)$, we call $(Y, B)$ a semicanonical pair.

**Lemma 1.** Every metric pair $(Y, B)$ is semicanonical.

**Proof.** Let $d$ be a metric for $Y$. For each $y \in Y - B$, let $V_y = \{x \in Y | d(x, y) < \frac{1}{2}d(y, B)\}$. Then the collection $\{V_y | y \in Y - B\}$ is a semicanonical cover for $(Y, B)$.

**Lemma 2.** Let $\mathcal{U}$ be a semicanonical cover for a pair $(Y, B)$, and let $C \subseteq B$. If $U$ is a neighborhood of $C$ in $Y$, then there exists a neighborhood $W$ of $C$ in $Y$ such that $\text{st}(W, \mathcal{U}) \subseteq U$.

**Proof.** Each $x \in C$ has a neighborhood $W_x$ in $Y$ such that $\text{st}(W_x, \mathcal{U}) \subseteq U$. Take $W = \bigcup_{x \in C} W_x$.

Suppose $f : X \rightarrow Y$ is a continuous closed surjection. A set $T \subseteq X$ is said to be saturated if $f^{-1}(f(T)) = T$. If $U \subseteq X$ is a neighborhood of a saturated set $T$, then there exists a saturated neighborhood $V$ of $T$ such that $V \subseteq U$. {Proof: Take $V = f^{-1}(Y - f(X - U))$.}

**Lemma 3.** Let $f : X \rightarrow Y$ be a continuous closed surjection, and suppose that $B$ is a closed subset of $Y$. If $(X, f^{-1}(B))$ is a semicanonical pair, then $(Y, B)$ is a semicanonical pair.

**Proof.** Let $A = f^{-1}(B)$, and let $\mathcal{U}$ be a semicanonical cover for $(X, A)$. For each $y \in Y - B$, let $G_y$ be a saturated neighborhood of $f^{-1}(y)$ in $X$ such that

$$(1) \quad G_y \subseteq \text{st}(f^{-1}(y), \mathcal{U}).$$

We shall show that

$$\mathcal{G} = \{f(G_y) | y \in Y - B\}$$

is a semicanonical cover for $Y - B$. Since $f$ is a closed surjection and since $G_y$ is open and saturated, $f(G_y)$ is open in $Y$, and it is obvious that

$$Y - B = \bigcup f(G_y) | y \in Y - B\}.$$

It remains to show that if $b \in B$ and if $U$ is a neighborhood of $b$ in $Y$, then there exists a neighborhood $U_0$ of $b$ such that $\text{st}(U_0, \mathcal{G}) \subseteq U$. By Lemma 2, $f^{-1}(b)$ has a neighborhood $W$ in $X$ such that $\text{st}(W, \mathcal{U}) \subseteq f^{-1}(U)$. Since $f$ is a continuous closed surjection, we may choose $W$ to be saturated. Similarly, there exists a saturated neighborhood $W_0$ of $f^{-1}(b)$ in $X$ such that $\text{st}(W_0, \mathcal{U}) \subseteq W$. The set $U_0 = f(W_0)$ is a neighborhood of $b$ in $Y$. We claim that $\text{st}(U_0, \mathcal{G}) \subseteq U$. For suppose that
Then, since \( G_y \) and \( W_0 \) are saturated and \( f \) is surjective,

\[
  (2) \quad G_y \cap W_0 \neq \emptyset.
\]

By (1) and (2), there exist a point \( x \in f^{-1}(y) \) and a \( V \in \mathcal{U} \) such that \( x \in V \) and \( V \cap W_0 \neq \emptyset \). Since \( st(W_0, \mathcal{U}) \subseteq W \),

\[
  (3) \quad V \subseteq W.
\]

Because \( W \) is saturated and \( x \in f^{-1}(y) \cap V \), it follows from (3) that \( f^{-1}(y) \subseteq W \). Since \( st(W, \mathcal{U}) \subseteq f^{-1}(U) \), we have in particular

\[
  (4) \quad st(f^{-1}(y), \mathcal{U}) \subseteq f^{-1}(U).
\]

It follows from (4) and (1) that \( f(G_y) \subseteq U \). Therefore, \( st(U_0, g) \subseteq U \), and the proof is complete.

3. Proof of Theorem B. Suppose \( f: M \to XX Y \) is a continuous closed surjection, where \( M \) is metrizable. Since \( XX F \) is homeomorphic to a closed subset of \( XX Y \), it follows that \( XX Y \) is the image of a metric space under a continuous closed surjection. Consequently, if \( y^* \) is an accumulation point of \( XX F \), then there exists a sequence \( y_1, y_2, \ldots \) in \( XX Y - \{ y^* \} \) such that

\[
  (1) \quad \lim_{n \to \infty} y_n = y^*.
\]

We shall show that \( XX F \) is first-countable. Let \( x \in XX F \). By Lemmas 1 and 3, \( (XX Y, XX Y - \{ y^* \}) \) is a semicanonical pair; let \( \mathcal{V} \) be a semicanonical cover for it. For each positive integer \( n \), choose \( V_n \in \mathcal{V} \) such that \( (x, y_n) \in V_n \). Define \( U_n = \pi(V_n) \), where \( \pi: XX Y \to XX X \) is the coordinate projection. We claim that \( \{ U_n \mid n \geq 1 \} \) is a basis for the neighborhoods of \( x \) in \( XX X \). To show this, let \( U \) be a neighborhood of \( x \) in \( XX X \), and let \( W \) be a neighborhood of \( (x, y^*) \) in \( XX Y \) such that \( st(W, \mathcal{V}) \subseteq \pi^{-1}(U) \). It follows from (1) that there exists an integer \( n \) such that \( W \cap V_n \neq \emptyset \). Therefore \( V_n \subseteq \pi^{-1}(U) \), and this implies that \( U_n \subseteq U \).

It follows that \( XX X \) is first-countable.

Similarly, \( YY Y \) is first-countable. Therefore \( XX XYY \) is first-countable, and, by Theorem A, metrizable.

Corollary. Let \( f: M \to XX Y \) be a continuous closed surjection, where \( M \) and \( YY Y \) are metrizable and \( YY Y \) is not discrete. Then \( XX Y \) is metrizable.

Proof. If \( XX X \) is discrete, then it is metrizable, and therefore \( XX Y \) is metrizable. If \( XX X \) is not discrete, the result follows from Theorem B.
REFERENCES


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