A NOTE ON CLOSED MAPS AND METRIZABILITY

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1. **Introduction.** Investigating conditions under which quotients of metric spaces are metrizable, Stone [3] and Morita and Hanai [2] independently obtained the following result.

THEOREM A. *If a topological space Y is the closed, continuous image of a metric space, and if Y satisfies the first axiom of countability, then Y is metrizable.*

The hypothesis of first-countability cannot be dropped. For example, if Y is the quotient space obtained from the real line R by identifying the integers to a point, then the natural projection

$$\pi: R \to Y$$

is closed and continuous, but Y is not metrizable. Notice that, although π is closed, the product map

$$\pi \times \pi : R \times R \to Y \times Y$$

is not closed. (For example, the set $\{(n, 1/n) \mid n=2, 3, 4, \cdots\}$ is closed in $R \times R$ but its image in $Y \times Y$ is not closed.) In fact, by Theorem B below, $Y \times Y$ is not the closed, continuous image of any metric space.

THEOREM B. *If X and Y are nondiscrete topological spaces and if $X \times Y$ is the closed, continuous image of a metric space, then $X \times Y$ is metrizable.*

We shall prove Theorem B, using Theorem A.

2. **Semicanonical covers.** Let $\mathcal{V}$ be a collection of subsets of a set S. For each $W \subset S$, we define the *star* of W with respect to $\mathcal{V}$ by

$$\mathrm{st}(W, \mathcal{V}) = \bigcup\{V \in \mathcal{V} \mid W \cap V \neq \varnothing\}.$$

A *pair* (Y, B) is a topological space Y together with a closed subset B. If Y is metrizable, then (Y, B) is called a *metric pair*.

Let (Y, B) be a pair. As in [1], we call a collection $\mathcal{V} = \{V_\alpha\}$ of open subsets of Y a *semicanonical cover* for (Y, B) if

(1) $\bigcup_\alpha V_\alpha = Y - B$, and

(2) for each $b \in B$ and each neighborhood[1] U of b in Y there exists

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[1] All neighborhoods are open.

a neighborhood W of b in Y such that $\mathrm{st}(W, \mathcal{V}) \subset U$.

If a semicanonical cover exists for (Y, B), we call (Y, B) a *semicanonical pair.*

LEMMA 1. *Every metric pair (Y, B) is semicanonical.*

PROOF. Let d be a metric for Y. For each $y \in Y - B$, let $V_y = \{x \in Y \mid d(x, y) < \frac{1}{2}d(y, B)\}$. Then the collection $\{V_y \mid y \in Y - B\}$ is a semicanonical cover for (Y, B).

LEMMA 2. *Let $\mathcal{V}$ be a semicanonical cover for a pair (Y, B), and let $C \subset B$. If U is a neighborhood of C in Y, then there exists a neighborhood W of C in Y such that* $\mathrm{st}(W, \mathcal{V}) \subset U$.

PROOF. Each $x \in C$ has a neighborhood W_x in Y such that $\mathrm{st}(W_x, \mathcal{V}) \subset U$. Take $W = \bigcup_{x \in C} W_x$.

Suppose $f: X \to Y$ is a continuous closed surjection. A set $T \subset X$ is said to be *saturated* if $f^{-1}f(T) = T$. If $U \subset X$ is a neighborhood of a saturated set T, then there exists a saturated neighborhood V of T such that $V \subset U$. {Proof: Take $V = f^{-1}(Y - f(X - U))$.}

LEMMA 3. *Let $f: X \to Y$ be a continuous closed surjection, and suppose that B is a closed subset of Y. If $(X, f^{-1}(B))$ is a semicanonical pair, then (Y, B) is a semicanonical pair.*

PROOF. Let $A = f^{-1}(B)$, and let $\mathcal{V}$ be a semicanonical cover for (X, A). For each $y \in Y - B$, let G_y be a saturated neighborhood of $f^{-1}(y)$ in X such that

$$G_y \subset \mathrm{st}(f^{-1}(y), \mathcal{V}). \tag{1}$$

We shall show that

$$\mathcal{G} = \{f(G_y) \mid y \in Y - B\}$$

is a semicanonical cover for $Y - B$. Since f is a closed surjection and since G_y is open and saturated, $f(G_y)$ is open in Y, and it is obvious that

$$Y - B = \bigcup \{f(G_y) \mid y \in Y - B\}.$$

It remains to show that if $b \in B$ and if U is a neighborhood of b in Y, then there exists a neighborhood U_0 of b such that $\mathrm{st}(U_0, \mathcal{G}) \subset U$. By Lemma 2, $f^{-1}(b)$ has a neighborhood W in X such that $\mathrm{st}(W, \mathcal{V}) \subset f^{-1}(U)$. Since f is a continuous closed surjection, we may choose W to be saturated. Similarly, there exists a saturated neighborhood W_0 of $f^{-1}(b)$ in X such that $\mathrm{st}(W_0, \mathcal{V}) \subset W$. The set $U_0 = f(W_0)$ is a neighborhood of b in Y. We claim that $\mathrm{st}(U_0, \mathcal{G}) \subset U$. For suppose that

$$f(G_y) \cap U_0 \neq \emptyset \quad \text{for some } y \in Y - B.$$

Then, since G_y and W_0 are saturated and f is surjective,

$$G_y \cap W_0 \neq \emptyset. \tag{2}$$

By (1) and (2), there exist a point $x \in f^{-1}(y)$ and a $V \in \mathcal{V}$ such that $x \in V$ and $V \cap W_0 \neq \emptyset$. Since $\mathrm{st}(W_0, \mathcal{V}) \subset W$,

$$V \subset W. \tag{3}$$

Because W is saturated and $x \in f^{-1}(y) \cap V$, it follows from (3) that $f^{-1}(y) \subset W$. Since $\mathrm{st}(W, \mathcal{V}) \subset f^{-1}(U)$, we have in particular

$$\mathrm{st}(f^{-1}(y), \mathcal{V}) \subset f^{-1}(U). \tag{4}$$

It follows from (4) and (1) that $f(G_y) \subset U$. Therefore, $\mathrm{st}(U_0, \mathcal{G}) \subset U$, and the proof is complete.

3. **Proof of Theorem B.** Suppose $f: M \to X \times Y$ is a continuous closed surjection, where M is metrizable. Since Y is homeomorphic to a closed subset of $X \times Y$, it follows that Y is the image of a metric space under a continuous closed surjection. Consequently, if y^* is an accumulation point of Y, then there exists a sequence $y_1, y_2, \cdots$ in $Y - \{y^*\}$ such that

$$\lim_{n \to \infty} y_n = y^*. \tag{1}$$

We shall show that X is first-countable. Let $x \in X$. By Lemmas 1 and 3, $(X \times Y, X \times \{y^*\})$ is a semicanonical pair; let $\mathcal{V}$ be a semicanonical cover for it. For each positive integer n, choose $V_n \in \mathcal{V}$ such that $(x, y_n) \in V_n$. Define $U_n = \pi(V_n)$, where $\pi: X \times Y \to X$ is the coordinate projection. We claim that $\{U_n \mid n \geqq 1\}$ is a basis for the neighborhoods of x in X. To show this, let U be a neighborhood of x in X, and let W be a neighborhood of (x, y^*) in $X \times Y$ such that $\mathrm{st}(W, \mathcal{V}) \subset \pi^{-1}(U)$. It follows from (1) that there exists an integer n such that $W \cap V_n \neq \emptyset$. Therefore $V_n \subset \pi^{-1}(U)$, and this implies that $U_n \subset U$. It follows that X is first-countable.

Similarly, Y is first-countable. Therefore $X \times Y$ is first-countable, and, by Theorem A, metrizable.

COROLLARY. *Let $f: M \to X \times Y$ be a continuous closed surjection, where M and Y are metrizable and Y is not discrete. Then $X \times Y$ is metrizable.*

PROOF. If X is discrete, then it is metrizable, and therefore $X \times Y$ is metrizable. If X is not discrete, the result follows from Theorem B.

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