ON THE ENDOMORPHISM RING OF A SIMPLE MODULE OVER AN ENVELOPING ALGEBRA

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Let U be an associative algebra with identity over a field k and let M be a simple U-module. If the cardinality of k is greater than $\dim_k M$, then there is a rather simple argument (see [**1**]) showing that any element θ of $\mathrm{Hom}_U(M, M)$ is algebraic over k. Namely if θ is not algebraic, then M is a nonzero vector space over $k(\theta)$, so $\dim_k M \geqq \dim_k k(\theta) \geqq \mathrm{card}\, k$, the last inequality coming from the fact that the elements $(\theta-\lambda)^{-1}$ of $k(\theta)$ are linearly independent. In this paper we prove by a completely different argument that the same conclusion holds for algebras U having a filtration like a universal enveloping algebra but without any cardinality assumption on k. I am grateful to Victor Guillemin for motivating the problem.

THEOREM. *Let k be a field and let U be an associative algebra with identity over k endowed with an increasing filtration $F_0U \subset F_1U \subset \cdots$ by k-subspaces such that*

(i) $1 \in F_0U,\ F_pU \cdot F_qU \subset F_{p+q}U,\ U = \bigcup F_pU.$

(ii) $\mathrm{gr}\, U = \oplus F_pU/F_{p-1}U$ *is a finitely generated commutative k-algebra.*

If M is a simple U-module, then every element of $\mathrm{Hom}_U(M, M)$ *is algebraic over k.*

PROOF. The proof is based on the following result from algebraic geometry [**3**, SGA 60–61, Expose IV, Lemma 6.7].

GENERIC FLATNESS LEMMA. *If A is a noetherian integral domain, B is an A-algebra of finite type (A and B are both commutative), and N is a finitely generated B-module, then there is a nonzero element f of A such that N_f is free over A_f.*

Let $\theta \in \mathrm{Hom}_U(M, M)$; as M is simple $\mathrm{Hom}_U(M, M)$ is a skew-field so the subring $k[\theta]$ generated by θ is an integral domain. Regard M as a $k[\theta] \otimes_k U$ module in the obvious way, let m_0 be a nonzero element of M so that $Um_0 = M$, and filter M by

$$F_pM = k[\theta] \cdot F_pU \cdot m_0.$$

Then $\mathrm{gr}\, M = \oplus F_pM/F_{p-1}M$ is a finitely generated module over

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$B = k[\theta] \otimes_k \operatorname{gr} U$ which in turn is a finitely generated commutative algebra over $A = k[\theta]$ by hypothesis (ii). By the generic flatness lemma there is a nonzero element f of $k[\theta]$ such that $(\operatorname{gr} M)_f$ is free over $k[\theta]_f$. As localization is exact $(\operatorname{gr} M)_f = \operatorname{gr}(M_f)$ where M_f is filtered by $F_p(M_f) = (F_p M)_f$; this filtration exhausts M_f by hypothesis (i), so we conclude that M_f is free over $k[\theta]_f$. But as $\operatorname{Hom}_U(M, M)$ is a skew-field M is a vector space over the quotient field $k(\theta)$ of $k[\theta]$, so $M = M_f$ is a direct sum of copies of $k(\theta)$ as a module over $k[\theta]_f$. This means that $k[\theta]_f \simeq k(\theta)$ which can happen only if θ is algebraic over k. Q.E.D.

The theorem applies to the universal enveloping algebra of a finite dimensional Lie algebra, hence the results of Dixmier [1] are valid without the assumption uncountability of k. In particular we have the following.

THEOREM. *Let $\mathfrak{g}$ be a nilpotent Lie algebra of finite dimension over a field k of characteristic zero. If M is a simple $\mathfrak{g}$-module, then the annihilator of M is a maximal ideal in $U(\mathfrak{g})$.*

PROOF. If I is the annihilator of M, then the center of $U(\mathfrak{g})/I$ is isomorphic to a subring of $\operatorname{Hom}_{U(\mathfrak{g})}(M, M)$ and hence by the theorem is algebraic over k. This center is therefore a field so by Corollary 2.4 of [2], I is maximal. Q.E.D.

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