A NOTE ON COVER AND AVOIDANCE PROPERTIES IN SOLVABLE GROUPS

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In [1] Carter showed the existence within a finite solvable group of a conjugate class of nilpotent self normalizing subgroups. In [2] Gaschütz defined Carter's work in the more general setting of formation theory. In [1] Carter showed that his groups satisfy a cover and avoidance property with respect to suitable factors of G. This note will extend this result within the framework of formation theory.

All groups considered will be finite and solvable. All definitions appear in [1] or [2] and all notations will be standard save where explicit definitions are given. $\mathfrak{F}$ will be a formation whose $\mathfrak{F}$-groups may or may not exist. $G(\mathfrak{F})$ will be the unique normal subgroup of G minimal with respect to the property that $G/G(\mathfrak{F})\in\mathfrak{F}$.

DEFINITION. If $M<G$, a series $\langle 1\rangle=G_0\lhd G_1\lhd\cdots\lhd G_n=G$ is called an M-series of G if G_i is normalized by M and G_i/G_{i-1} is a nontrivial irreducible M-factor.

THEOREM 1. *If F is an $\mathfrak{F}$-group of G and $\{G_i\}$, $0\leqq i\leqq n$ is any F series of G then F covers G_i/G_{i-1} if and only if $FG_i/G_{i-1}\in\mathfrak{F}$.*

PROOF. Suppose $FG_i/G_{i-1}\in\mathfrak{F}$ and F avoids G_i/G_{i-1}. Since $(F\cap G_i)G_{i-1}=G_{i-1}$ we have $F\cap G_i<G_{i-1}$ and $F\cap G_i=F\cap G_{i-1}$. On the other hand since F is an $\mathfrak{F}$-group, $F<FG_i$ and $FG_i/G_{i-1}\in\mathfrak{F}$ we get $FG_{i-1}=FG_i$. Since G_{i-1} is proper in G_i we get a contradiction to $F\cap G_i=F\cap G_{i-1}$. Suppose $FG_i/G_{i-1}\notin\mathfrak{F}$ and F covers G_i/G_{i-1}. We have $FG_i=FG_{i-1}$. Thus $FG_i/G_{i-1}=FG_i/G_{i-1}\cong F/F\cap G_{i-1}\in\mathfrak{F}$. This is a contradiction.

What is more interesting is that this cover avoidance property actually characterizes the $\mathfrak{F}$-groups of G.

THEOREM 2.[1] *If G is finite solvable and $\{G_i\}$ is an M-series of G such that M covers G_i/G_{i-1} if and only if $MG_i/G_{i-1}\in\mathfrak{F}$ then M is an $\mathfrak{F}$-group of G.*

PROOF. The proof will be by induction on $(G: M)|G|$. If $(G: M)|G|=1$ then there is nothing to prove. Thus we may assume $(G: M)|G|>1$. If $M=G$ then by hypothesis M covers all factors and thus $MG_1/G_0=G\in\mathfrak{F}$. Thus M is an $\mathfrak{F}$-group of G. If $M\neq G$ then M does

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not cover all the factors of the series $\{G_i\}$. Choose t maximal so that M does not cover G_t/G_{t-1}. Then it follows that $G=MG_t$, $G_t \trianglelefteq G$, $G_{t-1} \trianglelefteq G$, $G/G_t \in \mathfrak{F}$ and $G/G_{t-1} \notin \mathfrak{F}$. In particular $MG_{t-1} < G$. It is easy to see that $\{\Lambda_i\}$ where $\Lambda_i = G_i \cap MG_{t-1}$ forms an M-series of MG_{t-1} satisfying the hypothesis of the theorem. Also the homomorphic images of $\{G_i\}$ show that MG_{t-1}/G_{t-1} satisfies the hypotheses of the theorem in G/G_{t-1}. If $|G_{t-1}| > 1$ then induction on M in MG_{t-1} and MG_{t-1}/G_{t-1} in G/G_{t-1} yields that M is an $\mathfrak{F}$-group in MG_{t-1} and MG_{t-1}/G_{t-1} is an $\mathfrak{F}$-group in G/G_{t-1}. By [2, Lemma 2.3], it follows that M is an $\mathfrak{F}$-group of G. Thus $|G_{t-1}| = 1$ and G_t is a minimal normal subgroup of G. It follows that M is maximal in G and $M \cap G_1 = \langle 1 \rangle$. Thus since $G/G_1 \in \mathfrak{F}$ and $G \notin \mathfrak{F}$ we have that G_1 is precisely $G(\mathfrak{F})$. It follows that M is an $\mathfrak{F}$-group of G.

To bring Carter's cover avoidance property under our theorem we prove

THEOREM 3. *Suppose $M<G$ and M is nilpotent. Let H/K be an irreducible M-factor of G. Then MH/K is nilpotent if and only if H/K is M-central.*

PROOF. If MH/K is nilpotent since H/K is minimal normal in MH/K we have that $H/K < Z(MH/K)$. Thus $[M, H] < K$ or H/K is M-central. If H/K is M-central then MH/K is the product of two normal nilpotent groups MK/K and H/K. Thus MH/K is nilpotent.

REFERENCES

1. R. Carter, *Nilpotent self normalizing subgroups of solvable groups*, Math. Z. **75** (1961), 136–139.

2. W. Gaschütz, *Zur theorie der endlichen auflösbaren Gruppen*, Math. Z. **80** (1963), 300–305.

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