ON THE ZEROS OF THE BERGMAN FUNCTION IN DOUBLY-CONNECTED DOMAINS

PAUL ROSENTHAL

The purpose of this note is to show that every doubly-connected Lu Qi-Keng domain in $C^1$ is pseudoconformally equivalent to a disc with the center deleted. This extends a result of M. Skwarczynski [4], who gave an example of a domain $C^1$ which is not a Lu Qi-Keng domain. (See definition below.) Our results indicate that, at least in this particular case, there exists a connection between the degree of connectivity of $D$ and zeros of the Bergman function. We use the notation $z = (z_1, z_2, \ldots, z^n)$ for a point in $D \subset C^n$ and $\bar{z}$ for $(\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_n)$. We denote

$$D^* = \{i | i \in D\}.$$ 

Definitions and theorems relating to the Bergman function can be found in [1]. In what follows, the Bergman function of the domain $D \subset C^n$ will be denoted by $K_D(z, \bar{z})$.

**Definition.** A domain $D \subset C^n$ is a Lu Qi-Keng domain if the equation $K_D(z, \bar{z}) = 0$ has no solution in $D \times D^*$ (see [4]).

**Theorem 1.** Let $D$ be the ring $0 < r < |z| < 1$. Then $D$ is not a Lu Qi-Keng domain.

**Proof.** As was shown by Zarankiewicz in [5], see also [1, p. 10],

$$K_D(z, \bar{z}) = \frac{1}{\pi |z|^2} \left[ \varphi \left( \log(z\bar{z}) ; \omega, \omega' \right) + \frac{\eta}{\omega} - \frac{1}{2\omega'} \right],$$

$\varphi$ is the Weierstrassian $\varphi$-function, $\omega = \pi i$, $\omega' = \log r$, $2\eta$ is the increment of the Weierstrassian $\zeta$-function related to the half-period $\omega$. (We note that since the first half-period $\omega = \pi i$, the value of the $\varphi$-function does not depend on the value chosen for $\log(z\bar{z})$.) Using the Legendre equation $\eta \omega' - \eta' \omega = \pi i/2$ (real($\omega'/i\omega$) > 0), (1) can be written as

$$K_D(z, \bar{z}) = \frac{1}{\pi |z|^2} \left[ \varphi(u; \omega, \omega') + \frac{\eta'}{\omega'} \right],$$

$u = \log(|z|^2)$. The function $e^u$ maps the period-parallellogram (points

---

1 This work was supported in part by Air Force contract AF 1047-66 and AEC contract 326-P22.
Re $u = 0$, log $r^2$ excluded) onto the $q$-ring $0 < r^2 < |q| < 1$, $q = z_2$. Since the doubly periodic function $\mathcal{G}(u)$ attains every value in the period-parallelogram exactly twice, the function

$$v'/\omega' + \mathcal{G}(u)$$

attains every value in the $q$-ring except for values attained when $u$ is in the segment $\Re u = 0$, $0 \leq \Im u \leq \pi$. Consider the boundary of the rectangle with vertices $0$, $\pi i$, $\log r + \pi i$, $\log r$, in the $u$-plane with counterclockwise orientation. On this boundary, $\mathcal{G}$ attains real values increasing monotonically from $-\infty$ to $+\infty$. The function (3) has the same property, and we conclude that the exceptional values of (3) form a closed segment $[-\infty, v'/\omega' + \mathcal{G}(\pi i)]$ on the real axis. We infer the Bergman function has a zero in $D \times D^*$ if and only if

$$v'/\omega' + \mathcal{G}(\pi i) < 0.$$  

We prove next that, for every $0 < r < 1$, (4) holds. Consider the new pair of primitive half-periods, $\tilde{\omega} = -\log r$, $\tilde{\omega}' = \pi i$. We then obtain

$$v'/\omega' + \mathcal{G}(\pi i) = \tilde{v}/\tilde{\omega} + \mathcal{G}(\pi i).$$

It is known that [2, p. 336],

$$\frac{\tilde{v}}{\tilde{\omega}} + \mathcal{G}(u) = -\pi^2 \left( \frac{1}{\omega^2} \right) + \sum_{m=1}^{\infty} \frac{h_{2m+2}}{(1 - h_{2m+2})^2} + \sum_{m=1}^{\infty} \frac{h_{2m+2}}{(1 - h_{2m+2})^2}.$$  

$$v = u/2\omega, \quad z = e^{\pi i}, \quad t = \omega'/\tilde{\omega}, \quad h = e^{\pi i} (\Im t > 0).$$

Since the right-hand side of (6) for $\tilde{\omega} = -\log r$, $\tilde{\omega}' = \pi i$, and $u = \pi i$ is negative for all $0 < r < 1$, (4) holds. This completes the proof of Theorem 1.

**Theorem 2.** Every doubly-connected Lu Qi-Keng domain in $C^1$ is pseudoconformally equivalent to a disc with the center deleted.

**Proof.** Let $D$ be a doubly-connected Lu Qi-Keng domain in $C^1$. It must be pseudoconformally equivalent to one of the three following domains:

1. a plane with the center deleted,
2. a ring $0 < r < |z| < 1$,
3. a disc with the center deleted.

However, the Bergman function for the domain $\{z|z \neq 0\}$ is identically zero, and the Bergman function for the ring possesses zeros by
Theorem 1. Since the class of Lu Qi-Keng domains is invariant under pseudoconformal transformations, (1) and (2) do not occur.

**Theorem 3.** For every $k \geq 3$, there exists a domain $D \subset \mathbb{C}^1$ of connectivity $k$ which is not a Lu Qi-Keng domain.

**Proof.** Consider a ring $R$ and let $z_0$, $t_0$ be such that $K_R(z_0, t_0) = 0$. We choose $k - 2$ distinct points $z_1, \ldots, z_{k-2}$ in $R$ different from $z_0$ and $t_0$. Consider a domain $R_m = R - \bigcup_{j=1}^{k-2} \{z \in R \text{ and } |z - z_j| \leq 1/m, m \text{ a positive integer}\}$. By the Ramadanov Theorem [3], the sequence $K_{R_m}(z, t_0)$ converges locally uniformly to $K_D(z, t_0)$ where $D = \bigcup_{m=1}^{\infty} R_m$. Since $K_D(z, t_0) = K_R(z, t_0)$, we conclude that for sufficiently large $m$, the degree of connectivity of $R_m$ is $k$, and by Hurwitz’s theorem the function $K_{R_m}(z, t_0)$ has a zero in $R_m$.

**References**


**Stanford University**