ON A RELATION BETWEEN ABSOLUTE ABEL AND ABSOLUTE RIESZ SUMMABILITY

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1.1. Let $\sum a_n$ be an infinite series, and let $\{\lambda_n\}$ be an arbitrary sequence of positive numbers tending to infinity with n such that $1 \leqq \lambda_1 < \lambda_2 < \cdots$. We write

$$A_\lambda^k(x) = \sum_{\lambda_n < x} (x - \lambda_n)^k a_n = \int_1^x (x - t)^k dA_\lambda(t),$$

$$A_\lambda^0(x) = A_\lambda(x) = \sum_{\lambda_n < x} a_n,$$

$$A_\lambda^k(x) = 0 \qquad \text{for } x \leqq 1 \quad \text{and} \quad k > -1.$$

We also write

$$B_\lambda^k(x) = \sum_{\lambda_n < x} (x - \lambda_n)^k \lambda_n a_n.$$

The given series $\sum a_n$ is said to be summable (R, λ, k) to the sum s if $\lim x^{-k} A_\lambda^k(x) = s$ as $x \to \infty$; the series is said to be absolutely Riesz summable with index m, or simply $|R, \lambda, k|_m$ summable [6] if

$$\int_1^\infty x^{m-1} \left| \frac{d}{dx} x^{-k} A_\lambda^k(x) \right|^m dx < \infty,$$

where $k > 0$, $m \geqq 1$ and $k > 1 - 1/m$.

We say that the given series is summable $|R, \lambda, k, \gamma|_m$ [6] if

$$\int_1^\infty x^{m\gamma + m - 1} \left| \frac{d}{dx} x^{-k} A^k(x) \right|^m dx < \infty,$$

where $k > 1 - 1/m$, $k > \gamma - 1$ and γ is a real number. It is evident that $|R, \lambda, k, 0|_m$ summability is the same as $|R, \lambda, k|_m$ summability and $|R, \lambda, k|_1$ summability is the usual absolute Riesz summability denoted by $|R, \lambda, k|$.

The series $\sum a_n$ is said to be summable $|A, \lambda|_m$, $m \geqq 1$, if the series $f(x) = \sum a_n \exp[-\lambda_n x]$ converges for $x > 0$ and

$$\int_0^\infty (1 - e^{-x})^{m-1} |f'(x)|^m dx < \infty,$$

[9, Theorem 2].

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Summability $|A, \lambda|_m$ can be further generalized by introducing a real parameter γ as follows:

For $m \geqq 1$ and γ a real number, the series $\sum a_n$ is said to be summable $|A, \lambda, \gamma|_m$ if

(i) $f(x) = \sum a_n \exp[-\lambda_n x]$ converges for $x > 0$, and

(ii) $\int_0^\infty (1-e^{-x})^{m-m\gamma-1} |f'(x)|^m dx < \infty$.

It is obvious that summability $|A, \lambda, 0|_m$ is the same as summability $|A, \lambda|_m$ and $|A, \lambda|_1$ summability is the same as generalized absolute Abel summability denoted by $|A, \lambda|$. ($|A, \lambda|$ summability is defined implicitly in [10].)

1.2. In [8] we proved the following theorem.

THEOREM A. *If* (i) $\sum a_n$ *is summable* $|R, \lambda, k|_m$ *and* (ii) *the series* $\sum a_n \exp[-\lambda_n x]$ *converges for all* $x > 0$ *to the sum* $f(x)$, *then* (iii) $\sum a_n$ *is summable* $|A, \lambda|_m$.

In the special case $\lambda_n = n$, it follows from the results of Borwein [1] and Flett [3] that (i) implies (iii); so that, in this case, (i) alone implies (iii), but this is not true for arbitrary λ_n (see Remark below).

2.1. We prove the following theorems:

THEOREM 1. *If* (i) $\sum a_n$ *is summable* $|R, \lambda, k, \gamma|_m$, $m \geqq 1$, $k > 1 - 1/m$, $0 \leqq \gamma \leqq 1 - 1/m$ *and* (ii) *the series* $\sum a_n \exp[-\lambda_n x]$ *converges for all* $x > 0$ *to the sum* $f(x)$, *then* (iii) $\sum a_n$ *is summable* $|A, \lambda, \gamma|_m$.

THEOREM 2. *For* $0 < \gamma \leqq 1 - 1/n$, $k > 2 - 1/m$, $m > 1$, *summability* $|R, \lambda, k, \gamma|_m$ *of* $\sum a_n$ *implies its* $|A, \lambda, \gamma|_m$ *summability if and only if* $\liminf_{n\to\infty} \exp[\lambda_n x](\lambda_{n+1} - \lambda_n) > 0$ *for any* $x > 0$.

2.2. We require the following lemmas for the proofs of our theorems.

LEMMA 1 [5]. *If* $k > 0$, *then*

$$\frac{d}{dx}\left(x^{-k} A_\lambda^k(x)\right) = k x^{-k-1} B_\lambda^{k-1}(x).$$

LEMMA 2 [4, p. 27]. *If* $k > -1$, $p > 0$, *then*

$$A_\lambda^{k+p}(x) = M \int_1^x (x-t)^{p-1} A_\lambda^k(t)\,dt.$$

Throughout this paper M denotes a positive constant, not necessarily the same at every occurrence and m' is given by $1/m + 1/m' = 1$ for $m > 1$.

LEMMA 3 [7]. *Let* $\gamma>\mu$, $m>p\geqq 1$. *If* $\sum a_n$ *is summable* $|R, \lambda, k, \gamma|_m$, *then it is also summable* $|R, \lambda, k, \mu|_p$.

LEMMA 4 [4, p. 39]. *If* $\sum a_n \exp[-\lambda_n x]$ *converges for* $x>0$ *to the sum* $f(x)$, *then for* $k\geqq 0$,

$$f(x) = Mx^{k+1}\int_0^\infty A_\lambda^k(t)e^{-xt}dt.$$

LEMMA 5. *Let* $h>0$, $\gamma\geqq 0$, $m\geqq 1$. *If* $\sum a_n$ *is summable* $|R, \lambda, h, \gamma|_1$, *then it is also summable* $|R, \lambda, k, \gamma|_m$ *for* $k>h+1-1/m$, $h>\gamma-1$.

PROOF OF LEMMA 5. On account of Lemma 1, we need to show that

$$I = \int_1^\infty x^{m\gamma-mk-1}|B_\lambda^{k-1}(x)|^m dx < \infty,$$

given that

$$J = \int_1^\infty x^{\gamma-h-1}|B_\lambda^{h-1}(x)|^m dx < \infty.$$

Let $k=h+\delta$, where $\delta>1-1/m$.

By Lemma 2, we have

$$B_\lambda^{k-1}(x) = M\int_1^x (x-t)^{\delta-1}B_\lambda^{h-1}(t)dt.$$

Thus for $m>1$

$$|B_\lambda^{k-1}(x)| \leqq M\int_1^x (x-t)^{\delta-1}|B_\lambda^{h-1}(t)|dt$$

$$= M\int_1^x (x-t)^{\delta-1}t^{(h+1-\gamma)/m'}t^{-(h+1-\gamma)/m'}|B_\lambda^{h-1}(t)|^{1/m'}|B_\lambda^{h-1}(t)|^{1/m}dt.$$

Applying Holder's inequality we get

$$|B_\lambda^{k-1}(x)|^m \leqq M\left\{\int_1^x (x-t)^{m(\delta-1)}t^{(h+1-\gamma)(m-1)}|B_\lambda^{h-1}(t)|dt\right\}$$

$$\cdot\left\{\int_1^x t^{\gamma-h-1}|B_\lambda^{h-1}(t)|dt\right\}^{m-1},$$

$$\leqq M\int_1^x (x-t)^{m(\delta-1)}t^{(h+1-\gamma)(m-1)}|B_\lambda^{h-1}(t)|dt,$$

since

$$\int_1^x t^{\gamma-h-1} \mid B_\lambda^{h-1}(t) \mid dt \leqq J < \infty.$$

Therefore,

$$I \leqq M \int_1^\infty x^{m\gamma-mh-m\delta-1} dx \int_1^x (x-t)^{m\delta-m} t^{(h+1-\gamma)(m-1)} \mid B_\lambda^{h-1}(t) \mid dt$$

$$= M \int_1^\infty t^{(h+1-\gamma)(m-1)} \mid B_\lambda^{h-1}(t) \mid dt \int_t^\infty (x-t)^{m\delta-m} x^{m\gamma-mh-m\delta-1} dx$$

$$= M \int_1^\infty t^{\gamma-h-1} \mid B_\lambda^{h-1}(t) \mid dt = MJ < \infty.$$

The case $m=1$ is similar.

LEMMA 6. *If $\sum a_n$ is summable $|R, \lambda, k, \gamma|_m$, then it is also summable $|R, \lambda, h, \gamma|_m$ for $k>h$.*

The proof of Lemma 6 is similar to that of Lemma 3 in [7] and is therefore omitted.

2.3. PROOF OF THEOREM 1. In virtue of (ii),

$$f'(x) = -Mx^{k+1} \int_0^\infty B_\lambda^k(t) e^{-xt} dt.$$

Thus

$$\int_0^\infty (1-e^{-x})^{m-m\gamma-1} \mid f'(x) \mid^m dx$$

$$\leqq M \int_0^\infty (1-e^{-x})^{m-m\gamma-1} x^{mk+m} dx$$

$$\cdot \left\{ \int_0^\infty \mid B_\lambda^k(t) \mid^m e^{-xt} dt \right\} \left\{ \int_0^\infty e^{-xt} dt \right\}^{m-1}$$

$$= M \int_0^\infty \mid B_\lambda^k(t) \mid^m dt \int_0^\infty \left(\frac{1-e^{-x}}{x} \right)^{m-m\gamma-1} x^{mk+m-m\gamma} e^{-xt} dt$$

$$\leqq M \int_0^\infty x^{m\gamma-mk-m-1} \mid B_\lambda^k(t) \mid^m dt$$

$$< \infty,$$

by Lemma 1 and Lemma 6 with $h=k+1$.

PROOF OF THEOREM 2. Suppose

$$(1) \qquad \sum a_n \quad \text{is summable} \quad |R, \lambda, k, \gamma|_m$$

and

$$(2) \qquad \liminf_{n\to\infty} \exp[\lambda_n x](\lambda_{n+1} - \lambda_n) > 0 \qquad \text{for any } x > 0.$$

By Lemma 3, it is clear that $|R, \lambda, k, \gamma|_m$ summability of $\sum a_n$ implies its $|R, \lambda, k|$ summability, which in turn implies its (R, λ, k) summability. If x_k $(k \geqq 0)$ denotes the abscissa of (R, λ, k) summability of the Dirichlet series

$$(3) \qquad f(x) = \sum a_n \exp[-\lambda_n x],$$

then the assertion that $\sum a_n$ is summable (R, λ, k) implies $x_k \leqq 0$; while the convergence of (3) for all $x > 0$ will follow if $x_0 \leqq 0$. It therefore follows from Theorem 3.52 and Corollary 3.75 of [2] that the hypothesis of Corollary 3.75 is sufficient to ensure the convergence of (3) for all $x > 0$ if $\sum a_n$ is summable (R, λ, k). The hypothesis of Corollary 3.75 of [2], while expressed in a somewhat different form, may easily be seen to be equivalent to condition (2). Thus the sufficiency part of Theorem 2 is given by Theorem 1.

NECESSITY. Suppose condition (2) on $\{\lambda_n\}$ is not satisfied, then for some $x_0 > 0$,

$$\liminf_{n\to\infty} \exp[\lambda_n x_0](\lambda_{n+1} - \lambda_n) = 0.$$

We can find a sequence $\{n_m\}$ of positive integers such that $n_{m+1} \geqq n_m + 2$ and such that

$$(4) \qquad \sum_{m=0}^{\infty} \exp[\lambda(n_m)x_0][\lambda(n_m + 1) - \lambda(n_m)] < \infty,$$

where we write $\lambda(n)$ for λ_n whenever n is replaced by an expression which itself involves suffixes.

Define

$$\begin{aligned} a_n &= \exp[\lambda(n_m)x_0] && (n = n_m), \\ &= -\exp[\lambda(n_m)x_0] && (n = n_m + 1), \\ &= 0 \quad \text{otherwise}; \end{aligned}$$

then $f(x) = \sum a_n \exp[-\lambda_n x]$ clearly diverges for $x \leqq x_0$. Thus $\sum a_n$ is not summable $|A, \lambda, \gamma|_m$. On the other hand, the series $\sum a_n$ is summable $|R, \lambda, k, \gamma|_m$. To prove this, on account of Lemma 5, it is sufficient to prove that $\sum a_n$ is summable $|R, \lambda, 1, \gamma|_1$.

$$\int_1^\infty u^\gamma \left| \frac{d}{du} u^{-1} A_\lambda^1(u) \right| du$$

$$= M \sum_{m=0}^{\infty} \int_{\lambda(n_m)}^{\lambda(n_m+1)} u^\gamma \left| \frac{d}{du} u^{-1} A_\lambda^1(u) \right| du$$

$$\leqq M \sum_{m=0}^{\infty} \lambda^\gamma(n_m+1) \int_{\lambda(n_m)}^{\lambda(n_m+1)} \left| \frac{d}{du} u^{-1} A_\lambda^1(u) \right| du$$

$$= M \sum_{m=0}^{\infty} \lambda^\gamma(n_m+1) \int_{\lambda(n_m)}^{\lambda(n_m+1)} u^{-2} \left| B_\lambda^0(u) \right| du$$

$$= M \sum_{m=0}^{\infty} \exp[\lambda(n_m)x_0]\lambda^{\gamma-1}(n_m+1)[\lambda(n_m+1) - \lambda(n_m)]$$

$$< \infty,$$

by (4), since $\gamma < 1$.

This completes the proof of the theorem.

REMARK. It is clear that the above series $\sum a_n$ is summable $|R, \lambda, k, 0|_m$, i.e. summable $|R, \lambda, k|_m$ but not summable $|A, \lambda|_m$. Thus for arbitrary $\{\lambda_n\}$, in Theorem A (i) alone does not imply (iii).

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