A CHARACTERIZATION OF CELLULAR ARCS IN EUCLIDEAN 3-SPACE

FRANCIS D. LONERGAN

1. **Introduction.** As in [8], a topological space X is said to have a *contractible basis at a point x in X* if and only if there exists a basis $G_x = \{U_x\}$ of open neighborhoods of x with the property that each U_x is contractible (relative to some point) and furthermore $U_x - x$ is both pathwise and simply connected. The main result in this note is the following: An arc α in euclidean 3-space, E^3, is cellular if and only if the quotient space $X = E^3$ mod α obtained by collapsing α to a point has a contractible basis at the image $p(\alpha) = x$ (point) of α under the natural projection p of E^3 onto X.

Examples 1.1., 1.1* and 1.3 in [1], afford examples of noncellular arcs in E^3.

2. **Main result.** The definition of a combinatorial n-manifold can be found in [9, p. 290].

LEMMA 1. *If α is an arc in E^3, then α has arbitrarily small open neighborhoods with connected boundaries.*

PROOF. Let $\epsilon > 0$ be assigned and $U_\epsilon(\alpha)$ the open ϵ-neighborhood of α. Take a triangulation of E^3 of mesh $< \epsilon/3$ and let $N(\alpha, E^3)$ be the (connected) simplicial neighborhood of α relative to this triangulation. Take a regular neighborhood (connected), M^3, of $N(\alpha, E^3)$ lying within the open $\epsilon/3$-neighborhood of $N(\alpha, E^3)$. Then M^3 is a compact, connected, combinatorial 3-manifold with boundary B and $\alpha \subseteq$ interior of $M^3 = M^3 - B$. Suppose $B_1, \cdots, B_n$, $n \geqq 2$, are the components of B. Under a suitable successive barycenter subdivision of M^3, there exist vertices $a_2, \cdots, a_n$ in B_1, vertices $b_2, \cdots, b_n$ in $B_2, \cdots, B_n$ respectively, and a pairwise disjoint collection of simplicial arcs $\beta_2, \cdots, \beta_n$ (which do not meet α) from a_2 to $b_2, \cdots, a_n$ to b_n respectively such that for each j, $j = 2, \cdots, n$, $\beta_j - (a_j \cup b_j) \subseteq M^3 - B$. Since for $j = 2, \cdots, n$, β_j, a_j and b_j are geometrically collapsible, there exists (under a suitable subdivision of M^3) a pairwise disjoint collection of simplicial neighborhoods $N_2, \cdots, N_n$ of $\beta_2, \cdots, \beta_n$ respectively which are combinatorial 3-cells (3-elements) with the property that for $j = 2, \cdots, n$, $N_j \cap B_j = D_j$ and $N_j \cap B_1 = D_{j1}$ are combinatorial 2-cells, $N_j \cap B_i = \emptyset$ (empty set) for $i \neq 1$, $j \neq i$ and $\alpha \subseteq \mathring{M}_1^3 = M^3 - (B \cup \bigcup_{j=2}^n N_j)$ which is open in E^3. By a point-

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set argument one can show that the boundary, B^1, of $\mathring{M}_1^3$ is connected where, $B^1 = [B_1 - \bigcup_{j=2}^n \text{int } D_{j1}] \cup [\bigcup_{j=2}^n (B_j - \text{int } D_j)] \cup [\bigcup_{j=2}^n (N_j - \text{int } N_j)]$, with int D_{j1} = interior of D_{j1} relative to B_1; int D_j = interior of D_j relative to B_j; int N_j = interior of N_j relative to M^3. Thus, $\alpha \subseteq \mathring{M}_1^3 \subseteq \mathring{M}_1^3 \cup B^1 \subseteq M^3 \subseteq U_\epsilon(\alpha)$.

THEOREM 1. *An arc α in E^3 is cellular if and only if $X = E^3 \bmod \alpha$ has a contractible basis at $p(\alpha) = x$ (point).*

PROOF. If α is cellular, then X is homeomorphic to E^3 and so is locally euclidean [2]. Conversely, if X has a contractible basis at x then since x has arbitrarily small open neighborhoods with connected boundaries (since α has this property by Lemma 1) and X is a singular homology manifold with integers as coefficients [6], [7], it follows by the proof of Theorem II.3 in [8], that X is a (strong) homotopy manifold as defined in [3]. Hence by the proof of the Main Theorem in [3], α is a cellular arc.

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FLORIDA STATE UNIVERSITY