FINDING A BOUNDARY FOR A 3-MANIFOLD

L. S. HUSCH[1]

In [1] Browder, Levine and Livesay considered the following problem: "Given an open manifold M, when is it the interior of a compact manifold with boundary?" They were able to show that if dimension of $M \geqq 6$ and if M was 1-connected at infinity, then a necessary and sufficient condition was that the homology of M be finitely generated. Edwards [4] and Wall [10] showed that if dimension of M was 3 and if M was 1-connected at infinity then M was homeomorphic to the interior of a compact manifold. Siebenmann [9] obtained necessary and sufficient conditions when dimension of $M \geqq 6$. In this note, we prove the following.

THEOREM. *Let M be a connected, orientable 3-manifold with compact boundary and one end. The interior of M is homeomorphic to the interior of a compact 3-manifold if and only if there exists a positive integer N such that every compact subset of M is contained in the interior of a compact 3-manifold M′ with connected boundary such that*

1. $\pi_1(M-M')$ *is finitely generated*;
2. *genus* (bdry M') $\leqq N$;
3. *every contractible 2-sphere in* $M-M'$ *bounds a 3-cell.*

REMARKS. The referee has pointed out to the author that the boundary is unique by [12].

If the Poincaré Conjecture is true, hypothesis 3 is unnecessary. However if there is a counterexample to the Poincaré Conjecture it is possible to construct a counterexample to the theorem if 3 is not assumed. Whitehead's example [11] shows that 2 and 3 are not sufficient. It is unknown to the author whether 1 and 3 are sufficient. There is no loss of generality to assume that we are working in the piecewise linear category.

By van Kampen's Theorem [3, p. 71], it follows that $\pi_1(M)$ is finitely generated. Hence it follows that $M = \bigcup_{i=1}^{\infty} M_i$ such that for all $i = 1, 2, \cdots$

1. M_i *is a compact 3-manifold with connected boundary such that every contractible 2-sphere in* $M-M_i$ *bounds a 3-cell*;
2. $M_i \subseteq \text{int } M_{i+1}$ (*=interior* M_{i+1});
3. $\pi_1(M-M_i)$ *is finitely generated*;

Presented to the Society, August 29, 1967; received by the editors June 26, 1967.

[1] Research supported in part by National Science Foundation Grants GP-5458 and GP-6613.

4. $H_k(M, M_i)=0$ *for* $k=0, 1$;

5. *genus* (bdry M_i) $=n$ *where* n *is an integer such that if* $M=\bigcup_{i=1}^{\infty} M_i'$ *where* M_i' *satisfy* 1–4, *then genus* bdry $M_i' \geqq n$ *for all* i.

For all i, let $N_i=M_{i+1}-\text{int } M_i$. We have two cases to consider. If $n=0$, it follows from [6] that if $\pi_1(N_i)$ is trivial, then N_i is homeomorphic to the product of the 2-sphere with the interval. Hence if all but a finite number of N_i's have trivial fundamental group, we are through. If not, it follows from van Kampen's Theorem and [5, p. 192] that $\pi_1(M-M_1)$ is not finitely generated.

Let us now suppose that $n\geqq 1$. We make the following claim.

6. *The natural map* $\pi_1(\text{bdry } M_i)\rightarrow\pi_1(\bigcup_{k=i}^{\infty} N_k)$ *is a monomorphism for all but a finite number of* i'*s*.

Suppose that $\pi_1(\text{bdry } M_i)\rightarrow\pi_1(\bigcup_{k=i}^{\infty} N_k)$ is not a monomorphism. By the Loop Theorem [7], there exists a simple loop S on bdry M_i which is homotopically nontrivial on bdry M_i but S is contractible in $\bigcup_{k=i}^{\infty} N_k$. It follows from Dehn's Lemma [8] that $S=\text{bdry } D$ where D is a 2-cell such that $D\cap\text{bdry } M_i=S$. Let N be a regular neighborhood of D in $\bigcup_{k=i}^{\infty} N_k$ which meets bdry M_i in a regular neighborhood of S. Let $M_i'=M_i\cup N$.

We have two possibilities. If S does not separate bdry M_i, then M_i is a compact 3-manifold with connected boundary of genus $(n-1)$. Suppose S separates bdry M_i. Since $H_k(M, M_i)=0$, $k=0, 1$, then $H_k(M, M_i')=0$, $k=0, 1$. Hence $H_k(M-\text{int } M_i', \text{bdry } M_i')=0$, $k=0, 1$. Therefore $H_0(\text{bdry } M_i')=H_0(M-\text{int } M_i')$ and $M-\text{int } M_i'$ has two components, one of which is bounded, say B. Let $M_i''=M_i'\cup B$. Then M_i'' is a compact 3-manifold with connected boundary of genus $<n$.

If $\pi_1(\text{bdry } M_i)\rightarrow\pi_1(\bigcup_{k=i}^{\infty} N_k)$ is not a monomorphism for infinitely many i's then $M=\bigcup_{i=1}^{\infty} M_i^\alpha$ where genus (bdry M_i^α) $<n$ contradicting 5. Hence by reindexing we may assume

6'. *For* $i=1, 2, \cdots$, *the natural map* $\pi_1(\text{bdry } M_i)\rightarrow\pi_1(\bigcup_{k=i}^{\infty} N_k)$ *is a monomorphism*.

7. *There are only finitely many* i'*s such that there exists a surface* S *in* N_i *which separates* bdry M_i *from* bdry M_{i+1} *and which has genus* $<n$.

This claim follows from 5. Hence we have by reindexing

7'. *If* S *is a surface in* N_i, $i=1, 2, \cdots$, *which separates* bdry M_i *from* bdry M_{i+1}, *then genus* $S\geqq n$.

8. *For* $i=2, 3, \cdots$, *the natural map* $\pi_1(\text{bdry } M_i)\rightarrow\pi_1(\bigcup_{k=1}^{i-1} N_k)$ *is a monomorphism*.

The proof of 8 is by induction on i. Suppose $i=2$ and $\pi_1(\text{bdry } M_2)\rightarrow\pi_1(N_1)$ is not one-one. Then as in the proof of 6, there exists a 2-cell D in N_1 such that $D\cap\text{bdry } M_2=\text{bdry } D$ is a nontrivial loop on M_2.

Again we have two cases to consider depending upon whether bdry D separates bdry M_2.

If bdry D does not separate bdry M_2, then we can easily find a surface in the interior of N_1 separating bdry M_1 from bdry M_2 of genus $<n$. This contradicts 7′. Now consider the case when bdry D separates bdry M_2.

Let $\overline{N}$ be a regular neighborhood of D in N_1 such that $\overline{N} \cap$bdry M_2 is a regular neighborhood of bdry D in bdry M_2. Let $M_2' = M_2 - \text{int}(\overline{N} \cup N_2)$. Since M_1 can be chosen to contain the carriers of elements of $H_1(M)$, we have the following exact sequence

$$0 \to H_1(M, M_2') \to H_0(M_2') \to H_0(M) = 0.$$

But

$$H_1(M, M_2') = H_1(M - \text{int } M_2', \text{bdry } M_2').$$

Again,

$$H_1(\text{bdry } M_2') \to H_1(M - \text{int } M_2') \to H_1(M - \text{int } M_2', \text{bdry } M_2') \to H_0(\text{bdry } M_2') \to H_0(M - \text{int } M_2') = 0.$$

bdry D separates bdry M_2 implies that $H_0(\text{bdry } M_2') = Z$ (=integers). Since $H_1(M - \text{int } M_2', \text{bdry } M_2')$ maps onto Z and is free Abelian (for it is isomorphic to $H_0(M_2')$), $H_1(M - \text{int } M_2', \text{bdry } M_2')$ has rank greater than zero. Hence $H_0(M_2')$ has rank greater than zero and thereby the number of components of M_2' is greater than one. The boundary components of M_2' are A_1 and A_2 where $A_i \cap$bdry $M_2 \neq \varnothing$; in fact $A_1 \cap$bdry M_2 is separated from $A_2 \cap$bdry M_2 by bdry D. Since N_1 is a connected compact manifold, it follows that the component R_1 of M_2' which contains bdry M_1 also contains one of the A_i's, say A_1. By using the collar of A_1 in R, one can find a homeomorphic copy of A_1, say $\overline{A}_1$, such that $\overline{A}_1$ separates bdry M_1 from A_1. Since N_1 is connected, there is only one other component of M_2', say R_2, and bdry $R_2 = A_2$. Thus $\overline{A}_1$ separates bdry M_1 from bdry M_2 and genus $\overline{A}_1 < n$. This contradicts 7′. Hence $\pi_1(\text{bdry } M_2) \to \pi_1(N_1)$ is a monomorphism.

The induction argument is essentially the same as the case $i = 2$, except that the disk D may not lie in N_i; i.e., D may intersect $R = \bigcup_{k=2}^{i-1}$ bdry M_k. If so, put D into general position with respect to R, keeping bdry D fixed. Then $D \cap R$ is a finite collection of simple closed curves $\{S_j\}_{j=1}^n$. Pick an innermost S_j on D; suppose S_j bounds D_j on D and $S_j \subseteq$bdry M_k. Then D_j lies either in N_k or N_{k-1}. By either the induction hypothesis or 6′, S_j bounds a disk D_j' on bdry M_k. Replace D by $D' = (D - D_j) \cup D_j'$. By using the collar of bdry M_k we may "pop" D_j off bdry M_k, eliminating the singularity S_j without introducing any new singularities. After a finite number of these steps we finally get a disk $D''\subseteq N_i$ such that $D''\cap$bdry $N_i=$bdry D. We continue then as in the case $i=2$ to get a contradiction.

9. *If for $i=2, 3, \cdots$, the natural maps π_1(bdry $M_i)\to\pi_1(N_i)$ are epimorphisms, then the theorem is true.*

This claim follows from the following fact. If π_1(bdry $M_i)\to\pi_1(N_i)$ is an epimorphism, then by [2], N_i is homeomorphic to (bdry M_i) $\times[0, 1]$. We are left to consider the possibility that for infinitely many i's, π_1(bdry $M_i)\to\pi_1(N_i)$ is not onto. There is no loss of generality to assume that this occurs for all $i=2, 3, \cdots$. By van Kampen's theorem,

$$\pi_1\left(\bigcup_{i=1}^{k+1} N_i,\ p_{k+1}\right) = \pi_1\left(\bigcup_{i=1}^{k} N_i,\ p_{k+1}\right) *_{G_{k+1}} \pi_1(N_{k+1},\ p_{k+1})$$

i.e., if p_{k+1} is a point in bdry M_{k+1}, then the fundamental group of $\bigcup_{i=1}^{k+1} N_i$ is the free product of the fundamental groups of $\bigcup_{i=1}^{k} N_i$ and N_{k+1} with amalgamated subgroup $G_{k+1}=\pi_1$(bdry M_{k+1}, p_{k+1}). It follows from 6′ and 8 that the "natural map"

$$\phi\colon \pi_1\left(\bigcup_{i=1}^{k} N_i,\ p_{k+1}\right)\to \pi_1\left(\bigcup_{i=1}^{k+1} N_i,\ p_{k+1}\right)$$

is a monomorphism. Since π_1(bdry $M_{k+1}, p_{k+1})\to\pi_1(N_{k+1}, p_{k+1})$ is not onto, ϕ is not onto. Hence we may assume that $\pi_1(\bigcup_{i=1}^{k} N_i, p_1)$ is identified with a proper subgroup of $\pi_1(\bigcup_{i=1}^{k+1} N_i, p_1)$ with "identifying map" ϕ_k. Then $\pi_1(\bigcup_{i=1}^{\infty} N_i)$ is the direct limit of $\{\pi_1(\bigcup_{i=1}^{k} N_i), \phi_k\}$ and hence can be written as the infinite monotone union of proper subgroups which implies that $\pi_1(\bigcup_{i=1}^{\infty} N_i)=\pi_1(M-M_i)$ is infinitely generated. This contradiction establishes the theorem.

References

1. W. Browder, J. Levine and G. R. Livesay, *Finding a boundary for an open manifold,* Amer. J. Math. **87** (1965), 1017–1028.

2. E. M. Brown, *Unknotting in $M^2\times I$,* Trans. Amer. Math. Soc. **123** (1966), 480–505.

3. R. Crowell and R. H. Fox, *Introduction to knot theory,* Ginn, New York, 1963.

4. C. H. Edwards, Jr., *Open 3-manifolds which are simply connected at infinity,* Proc. Amer. Math. Soc. **14** (1963), 391–395.

5. A. Karass, W. Magnus and D. Solitar, *Combinatorial group theory,* Interscience, New York, 1966.

6. J. Milnor, *A unique decomposition theory for 3-manifolds,* Amer. J. Math. **84** (1962), 1–7.

7. C. D. Papakyriakopoulos, *On solid tori*, Proc. London Math. Soc. (3) **7** (1957), 281–299.

8. ———, *On Dehn's lemma and the asphericity of knots*, Ann. of Math. (2) **66** (1957), 1–26.

9. L. Siebenmann, *The obstruction to finding a boundary for an open manifold of dimension greater than five*, Doctoral Dissertation, Princeton Univ., Princeton, N. J., 1965.

10. C. T. C. Wall, *Open 3-manifolds which are 1-connected at infinity*, Quart. J. Math. Oxford Ser. **16** (1965) 263–268.

11. J. H. C. Whitehead, *A certain open manifold whose group is unity*, Quart. J. Math. Oxford Ser. **6** (1935) 268–279.

12. C. H. Edwards, Jr., *Concentricity in 3-manifolds*, Trans. Amer. Math. Soc. **113** (1964), 406–423.

Florida State University and
University of Georgia