EXTENDING ULM'S THEOREM WITHOUT GROUP THEORY[1]

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1. **Introduction.** In 1933 Ulm [6] showed that a countable reduced abelian p-group G was determined by a function F_G from ordinals to cardinals defined by $F_G(\alpha) = \dim p^\alpha G[p]/p^{\alpha+1}G[p]$, the Ulm invariants of G. In 1960 Kolettis [3] extended this result to direct sums of such groups. Hill [1] has recently improved Kolettis' work by showing that if G_i and H_i are countable reduced abelian p-groups such that $G = \sum_{i\in I} G_i$ and $H = \sum_{i\in I} H_i$ have the same Ulm invariants then there exists a partition of I into countable subsets I_α such that $\sum_{i\in I_\alpha} G_i$ and $\sum_{i\in I_\alpha} H_i$ have the same Ulm invariants. Ulm's theorem then implies that G and H are isomorphic.

Hill's work uses the fact that if G is a direct sum of countable reduced p-groups then the socle of G is decomposable, i.e. $G[p] = \sum_{\lambda<\lambda(G)} K_\lambda$ where every nonzero element of K_λ has height precisely λ in G. This is a nontrivial group theoretic property, whereas one would suspect that Hill's theorem holds on purely set theoretic grounds.

There is more than esthetics involved in eliminating the requirement of the decomposability of $G[p]$. Let $\mathcal{C}$ be the class of totally projective p-groups [4] of length less than $\Omega\omega$ where Ω and ω are, respectively, the first uncountable and the first infinite ordinals. Parker and Walker [5] have shown that two groups in $\mathcal{C}$ are isomorphic if and only if they have the same Ulm invariants. This generalizes Kolettis' theorem since $\mathcal{C}$ contains the class of direct sums of countable reduced p-groups [4]. In proving this generalization the following situation arises: $G = \sum_{i\in I} G_i$ and $H = \sum_{i\in I} H_i$ have the same Ulm invariants, G_i and H_i are totally projective with $|G_i|$, $|H_i| \leqq \aleph_1$ and it is necessary to partition I into subsets I_α such that $|I_\alpha| \leqq \aleph_1$ and $\sum_{i\in I_\alpha} G_i$ and $\sum_{i\in I_\alpha} H_i$ have the same Ulm invariants. Hill's procedure cannot be followed because the socles of G and H need not be decomposable. In fact [2], if $G[p]$ is decomposable then $p^\Omega G = \{0\}$. Again, such a partition of I, if it exists, should exist on purely set theoretic grounds. This is indeed the case and our purpose here is to establish the relevant set theoretic facts which immediately yield both this and the theorem of Hill.

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2. **The theorem.** We separate part of the construction into a lemma.

LEMMA. *Let m be an infinite cardinal number. Let f be a function from the set X to the cardinal numbers. Suppose*

$$f(x) = \sum_{\lambda\in I} f_\lambda(x) = \sum_{\lambda\in I} g_\lambda(x) \qquad \text{for all } x \in X$$

where $\sum_{x\in X} f_\lambda(x) \leqq m \geqq \sum_{x\in X} g_\lambda(x)$ *for all* $\lambda\in I$, *and* $m < |I|$. *If* $S\subseteq I$, $|S| < |I|$ *then there exists a set* $\bar{S}\supseteq S$ *such that*

$$\sum_{\lambda\in\bar{S}} f_\lambda(x) = \sum_{\lambda\in\bar{S}} g_\lambda(x), \qquad \sum_{\lambda\in I\setminus\bar{S}} f_\lambda(x) = \sum_{\lambda\in I\setminus\bar{S}} g_\lambda(x)$$

and

Case 1. $|I|$ *is not the sum of* $\aleph_0$ *smaller cardinals*: $|\bar{S}| < |I|$ *and if for some* $x\in X$, $f(x) < |I|$, $\sum_{\lambda\in\bar{S}} f_\lambda(x) \neq 0$ *and* $f_\lambda(x)$ *or* $g_\lambda(x) \neq 0$, *then* $\lambda\in\bar{S}$.

Case 2. $|I|$ *is a sum of* $\aleph_0$ *smaller cardinals*: $|\bar{S}| \leqq m|S|$.

PROOF. Construct $S=S_1\subseteq T_1\subseteq U_1\subseteq S_2\subseteq T_2\subseteq \cdots$ so that

(a) $\sum_{\lambda\in U_j} f_\lambda(x) \geqq \sum_{\lambda\in T_j} g_\lambda(x) \geqq \sum_{\lambda\in S_j} f_\lambda(x)$.

(b) Case 1: $|S_j| < |I|$. Case 2: $|S_j| \leqq m|S|$.

(c) $S_{j+1} = U_j \cup \{\lambda \mid \exists x\in X,\ \sum_{\lambda\in S_j} f_\lambda(x) \neq 0$ and $f_\lambda(x)$ or $g_\lambda(x)\neq 0$ and, Case 1: $f(x) < |I|$, Case 2: $f(x) \leqq m|S|\}$.

Let $\bar{S} = \bigcup S_j$. Since $\bar{S} = \bigcup S_j = \bigcup T_j = \bigcup U_j$ it follows from (a) that $\sum_{\lambda\in\bar{S}} f_\lambda(x) = \sum_{\lambda\in\bar{S}} g_\lambda(x)$. In Case 1, since $|S_j| < |I|$ then $|\bar{S}| < |I|$; in Case 2, since $|S_j| \leqq m|S|$ we have $|\bar{S}| \leqq m|S|$. Suppose $\sum_{\lambda\in I\setminus\bar{S}} f_\lambda(x) \neq \sum_{\lambda\in I\setminus\bar{S}} g_\lambda(x)$ for some $x\in X$. Since $f(x) = \sum_{\lambda\in\bar{S}} f_\lambda(x) + \sum_{\lambda\in I\setminus\bar{S}} f_\lambda(x) = \sum_{\lambda\in\bar{S}} g_\lambda(x) + \sum_{\lambda\in I\setminus\bar{S}} g_\lambda(x)$ and $\sum_{\lambda\in\bar{S}} f_\lambda(x) = \sum_{\lambda\in\bar{S}} g_\lambda(x)$ is, in Case 1, less than $|I|$ and, in Case 2, less than or equal to $m|S|$, we must have, in Case 1, that $f(x) < |I|$ and, in Case 2, that $f(x) \leqq m|S|$. Hence by (c) every λ such that $f_\lambda(x)$ or $g_\lambda(x) \neq 0$ must be in $\bar{S}$. But this implies that $\sum_{\lambda\in I\setminus\bar{S}} f_\lambda(x) = \sum_{\lambda\in I\setminus\bar{S}} g_\lambda(x) = 0$. The second condition in Case 1 follows immediately from (c).

THEOREM. *Let m be an infinite cardinal number. Let f be a function from the set X to the cardinal numbers such that*

$$f(x) = \sum_{\lambda\in I} f_\lambda(x) = \sum_{\lambda\in I} g_\lambda(x) \qquad \text{for all } x \in X,$$

where $\sum_{x\in X} f_\lambda(x) \leqq m \geqq \sum_{x\in X} g_\lambda(x)$ *for all* $\lambda\in I$. *Then there exists a partition of I into subsets S_α of cardinality* $\leqq m$ *such that*

$$\sum_{\lambda\in S_\alpha} f_\lambda(x) = \sum_{\lambda\in S_\alpha} g_\lambda(x).$$

PROOF. We show that if $|I| > m$ then such a partition can be found where $|S_\alpha| < |I|$. By induction we will have proved the theorem.

Initially well order $I = \{i_\alpha\}$, $\alpha < |I|$.

Case 1. Define sets $S_\alpha \subseteq I$ for ordinals $\alpha < |I|$ as follows: Let

$$S_\alpha = \left\{\overline{i_\beta \notin \bigcup_{\delta<\alpha} S_\delta \,\middle|\, \beta = \alpha \quad \text{or} \quad \exists\gamma > \beta,\, i_\gamma \in \bigcup_{\delta<\alpha} S_\delta}\right\}$$

where the "bar" is taken inside $I \backslash \bigcup_{\delta<\alpha} S_\delta$. Observe that $|S_\alpha| < |I|$ and $|\bigcup_{\delta<\alpha} S_\delta| = |I|$ only if $\bigcup_{\delta<\alpha} S_\delta = I$. Also,

$$\sum_{\lambda \in I \backslash \bigcup_{\delta<\alpha} S_\delta} f_\lambda(x) = \sum_{\lambda \in I \backslash \bigcup_{\delta<\alpha} S_\delta} g_\lambda(x),$$

for otherwise $f(x) < |I|$ and hence $f_\lambda(x) = g_\lambda(x) = 0$ for all $\lambda \in I \backslash \bigcup_{\delta<\alpha} S_\delta$ or for all $\lambda \in \bigcup_{\delta<\alpha} S_\delta$. Thus the construction makes sense and gives the desired partition.

In Case 2, let $\alpha_1 < \alpha_2 < \alpha_3 < \cdots$ be a sequence of ordinals with limit $|I|$. Define

$$S_j = \left\{\overline{i_\beta \notin \bigcup_{i<j} S_i \,\middle|\, \beta < \alpha_j}\right\}$$

where the "bar" is taken inside $I \backslash \bigcup_{i<j} S_i$. The S_j give the desired partition.

COROLLARY (HILL). *If $G = \sum_{\lambda \in I} A_\lambda$ and $H = \sum_{\lambda \in I} B_\lambda$ are reduced p-groups with the same Ulm invariants, $|A_\lambda| \leqq \aleph_0 \geqq |B_\lambda|$, then there is a partition of the set I into countable sets S_α such that*

$$\sum_{\lambda \in S_\alpha} A_\lambda \cong \sum_{\lambda \in S_\alpha} B_\lambda.$$

PROOF. Apply the theorem to the Ulm invariants of G and use Ulm's theorem.

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