ON A COMPLEMENT TO VALIRON'S TAUBERIAN THEOREM FOR THE STIELTJES TRANSFORM

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One of the classical abelian-tauberian theorems, due to Valiron [**14**], is

THEOREM A. *Let ψ be an increasing function such that $\psi(0)=0$ and*

$$F(x) = \int_0^\infty \frac{d\psi(t)}{x+t} \tag{1}$$

converges for $x>0$. Then

$$\psi(t) \sim t^\lambda L(t) \qquad (0 \leqq \lambda < 1, t \to \infty), \tag{2}$$

where L is a slowly-varying function, if and only if[2]

$$F(x) \sim (\pi\lambda/\sin \pi\lambda) x^{\lambda-1} L(x) \qquad (x \to \infty). \tag{3}$$

The term "slowly-varying" here is used in the sense of Karamata [**8**], and means that L is positive and satisfies

$$L(\sigma t)/L(t) \to 1 \qquad (t \to \infty) \tag{K}$$

for every $\sigma > 0$.

The relation between Theorem A, stated in terms of slowly-varying functions L, and Valiron's original form of the theorem which uses the notion of "proximate orders," is explained in [**7**].

Feller [**5**, p. 419] has pointed out the importance of observing that in such an abelian-tauberian statement, the relations (2) and (3) are equivalent to

$$\lim_{t\to\infty} \frac{\psi(\sigma t)}{tF(t)} = \frac{\sin \pi\lambda}{\pi\lambda} \sigma^\lambda \qquad (0 < \sigma < \infty). \tag{4}$$

That each of (2) and (3) implies (4) is the content of Theorem A, a standard tauberian theorem of some depth (cf. [**12**], [**6**]). The converse assertion, that (4) implies (2) and (3), is on the other hand quite obvious: for example, (4) implies that $\psi(\sigma t)/\psi(t) \to \sigma^\lambda$ $(t \to \infty)$ for every $\sigma > 0$, and this is clearly equivalent to (2). However, the following theorem shows that the truth of (4) for a *single* value of $\sigma(\sigma = 1$, say) is already sufficient to imply (2) and (3).

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[2] When $\lambda = 0$, $\pi\lambda/\sin \pi\lambda$ is to be replaced by its limit, $=1$.

THEOREM B. *Let ψ and F be as defined in Theorem* A. *Then the condition*

$$\alpha = \lim_{t\to\infty} \frac{\psi(t)}{tF(t)} \quad \textit{exists} \quad (\alpha > 0) \tag{5}$$

implies $\alpha \leqq 1$, *and* (2) *and* (3) *are true with* λ *determined by*

$$\alpha = \sin \pi\lambda/\pi\lambda \qquad (0 \leqq \lambda < 1). \tag{6}$$

Although Theorem A (as well as many similar tauberian theorems) is often presented with conditions on L other than (K), part of the significance of Theorem B is that it justifies Karamata's condition as the most natural one. (In this connection, cf. also Feller's discussion of slow variation in [**4**, p. 317].)

An early form of Theorem B appeared in my thesis (cf. [**11**, Corollary 1.1] for the statement and proof of an analogous result); however, that result was not sharp since I proved there only that (5) implies (2) and (3) with

$$L(t) = t^{\epsilon(t)} \qquad (\epsilon(t) \to 0 \text{ as } t \to \infty)$$

in place of (K).

More recently, Edrei and Fuchs [**3**] established the sharp form of Theorem B given above, together with a similar result involving a kernel different from the one in (1). Their proofs use an impressive array of ingenious and powerful techniques developed in [**3**] as well as in some of their earlier joint work.

Still more recently, Drasin [**1**] has succeeded in extending the method of Edrei and Fuchs to cover a wide class of convolution transforms

$$F(x) = \int_{-\infty}^{\infty} k(x - t)f(t)dt. \tag{7}$$

Thus if k is a strictly positive $L^1(-\infty, \infty)$ kernel behaving suitably at $\pm\infty$, f increases (or does not decrease too rapidly), and F is defined by (7), then Drasin's theorem asserts that the existence and positivity of $\lim_{x\to\infty} f(x)/F(x)$ implies $f(x) = e^{\lambda x}\phi(x)$, where $L(t) = \phi(\log t)$ satisfies (K). In particular, this general result contains Theorem B.

It is the purpose of this note to develop a new method of proving Theorem B which is short and quite transparent, and leads immediately to several refinements (Theorems B′ and C, discussed in §2). The present method is also capable of considerable generalization, but I confine myself here to the case of the Stieltjes transform (1) in order to present the main ideas as clearly as possible.

The proof of Theorem B to be given here also throws some light on the essential features of the "tauberian" (for lack of a better term) hypothesis (5). In particular, the first part of the proof involves reducing the problem to that of solving a certain integral equation; this in itself is typical of tauberian theorems (compare Wiener's remarks in [**16**, p. 50]). But a comparison of the difficulties encountered in dealing with the trivial integral equation corresponding to the classical theorem (3)⇒(2), and those involved here in proving (5) ⇒(2), suggests that in fact Theorem B lies deeper than Theorem A.

One further remark: the statement, as well as the present proof, of Theorem B contrast interestingly with the Paley-Wiener generalization of Mercer's theorem [**10**], especially in the dependence of this proof on the location of the "ones" of the Fourier transform of the kernel in (1).

1. **Proof of Theorem B.** An integration by parts transforms (1) into

$$F(x) = \int_0^\infty \frac{\psi(t)}{(t+x)^2}\,dt, \tag{1.1}$$

in view of the following simple remark (which we shall have occasion to use again): *If ϕ is a positive, increasing function such that*

$$\int_1^\infty \frac{d\phi(t)}{t} < \infty \quad \textit{or} \quad \int_1^\infty \frac{\phi(t)}{t^2}\,dt < \infty,$$

then

$$\phi(t) = o(t) \qquad (t \to \infty). \tag{A}$$

It is clear from (1.1) that $F(ux) \leqq F(x)$ for $u \geqq 1$. This observation together with (5) shows that

$$1 \leqq \liminf_{t\to\infty} \frac{\psi(ut)}{\psi(t)} \leqq \limsup_{t\to\infty} \frac{\psi(ut)}{\psi(t)} \leqq u \tag{1.2}$$

for all $u \geqq 1$. Fix $\sigma > 1$, and choose any sequence $t_n \to \infty$ such that

$$C = \lim_{n\to\infty} \frac{\psi(\sigma t_n)}{\psi(t_n)} \tag{1.3}$$

exists. Then the functions $g_n(u) = \psi(ut_n)/\psi(t_n)$ clearly increase and are uniformly bounded on any finite interval.

Applying the "selection principle" [15] successively on the intervals $0\leqq u\leqq u_0$ ($u_0=1, 2, \cdots$) in a standard way, we find a subsequence n_k such that g_{n_k} converges for all $u\geqq 0$ to an increasing function g, with

$$g(1) = 1, \qquad g(\sigma) = C. \tag{1.4}$$

Making a change of variables in (1.1) leads to

$$\frac{x\tau_k F(x\tau_k)}{\psi(x\tau_k)}\,\frac{\psi(x\tau_k)}{\psi(\tau_k)} = x\int_0^\infty \frac{\psi(u\tau_k)}{\psi(\tau_k)}\,\frac{du}{(u+x)^2} \qquad (\tau_k = t_{n_k}), \tag{1.5}$$

for any $x>0$ and all sufficiently large k. By Fatou's lemma, and (5), we deduce

$$g(x) \geqq \alpha x\int_0^\infty \frac{g(u)}{(u+x)^2}\,du. \tag{1.6}$$

In fact, *equality* holds in (1.6). To see this, we use the estimate

$$F(r) < \int_0^R \frac{\psi(t)}{(t+r)^2}\,dt + 4F(R) \qquad (r>0,\ R>0), \tag{1.7}$$

an immediate consequence of the simple relations

$$\int_R^\infty \frac{\psi(t)}{(t+r)^2}\,dt < \int_R^\infty \frac{\psi(t)}{t^2}\,dt < 4\int_R^\infty \frac{\psi(t)}{(t+R)^2}\,dt \leqq 4F(R).$$

In (1.7) set $r=x\tau_k$ and $R=sx\tau_k$, where $x>0$, $s>0$; then the left side of (1.5) is dominated by

$$x\int_0^{sx} \frac{\psi(u\tau_k)}{\psi(\tau_k)}\,\frac{du}{(u+x)^2} + \left(\frac{4}{s}\right)\frac{sx\tau_k F(sx\tau_k)}{\psi(sx\tau_k)}\,\frac{\psi(sx\tau_k)}{\psi(\tau_k)}$$

for all sufficiently large k. Letting $k\to\infty$, we use (5) again, and bounded convergence, to deduce

$$g(x) \leqq \alpha x\int_0^{sx} \frac{g(u)}{(u+x)^2}\,du + \frac{4}{s}\,g(sx).$$

Now let $s\to\infty$, and use (A) and (1.6) to obtain finally

$$g(x) = \alpha x\int_0^\infty \frac{g(u)}{(u+x)^2}\,du \qquad (x>0). \tag{1.8}$$

A solution of this equation which increases and satisfies $g(1)=1$ is given by $g(x)=x^\lambda$, with λ determined by (6). If this were the *only* such solution, then we would have necessarily $C=\sigma^\lambda$, by (1.4), and thus a comparison with (1.3) yields

$$\lim_{t\to\infty} \frac{\psi(\sigma t)}{\psi(t)} = \sigma^\lambda \qquad (0 < \sigma < \infty), \tag{1.9}$$

since in the above argument $\sigma(>1)$ was arbitrary. But (1.9) is obviously equivalent to (2).

To complete the proof of Theorem B, then, it is enough to show that the only *admissible* (i.e. increasing, positive) solutions of (1.8) are given by

$$g(x) = Ax^\lambda \qquad (A > 0), \tag{1.10}$$

where λ satisfies (6).

We first show that

$$\beta = \limsup_{x\to\infty} \frac{\log g(x)}{\log x} < 1 \tag{1.11}$$

holds for any admissible solution g of (1.8). Indeed, it is obvious from (1.8) and (A) that $\beta \leqq 1$, and if $\beta=1$, then an elementary argument [**11**, p. 208] shows that there exist sequences x_n, S_n such that

$$x_n \to \infty, \quad S_n/x_n \to \infty \qquad (n \to \infty) \tag{1.12}$$

and

$$g(u) \geqq (1 + o(1))(g(x_n)/x_n)u \qquad (x_n \leqq u \leqq S_n, \text{ to } n \to \infty).$$

Using this inequality in the estimand (1.6) yields

$$g(x_n) \geqq \alpha(1 + o(1))g(x_n) \int_{x_n}^{S_n} \frac{udu}{(u + x_n)^2} \qquad (n \to \infty)$$

and hence

$$1 \geqq \frac{\alpha}{2} \int_1^{S_n/x_n} \frac{tdt}{(t + 1)^2} \qquad (n \geqq n_0).$$

This inequality is impossible in view of (1.12) and the fact that $\alpha>0$, and thus (1.11) is true.

Making an exponential change of variable, equation (1.8) becomes

$$G(y) = \int_{-\infty}^{\infty} G(t)k(y - t)dt \qquad (G(y) = g(e^y)), \tag{1.13}$$

with

$$k(x) = \alpha(e^{x/2} + e^{-x/2})^{-2}.$$

A standard result in Fourier analysis [**13**, p. 305] asserts that any solution of (1.13) such that

$$G(y) = O(e^{a|y|}) \qquad (y \to \pm \infty) \tag{1.14}$$

for some $a<1$ must have the form

$$G(y) = \sum_{\nu} \sum_{p=1}^{q_\nu} A_{\nu,p} y^{p-1} \exp[-i\omega_\nu y], \tag{1.15}$$

where ω_ν runs through the sequence of zeros of $\hat{k}(\omega)-1$ such that $|\text{Im } \omega_\nu| \leqq a$, and where q_ν is the multiplicity of the zero ω_ν, the $A_{\nu,p}$ are constants and

$$\hat{k}(\omega) = \int_{-\infty}^{\infty} k(x)e^{ix\omega}dx = \alpha \int_0^{\infty} \frac{t^{i\omega}}{(t+1)^2}dt = \alpha \frac{i\pi\omega}{\sin(i\pi\omega)} \cdot$$

Clearly, there can only be a finite number of solutions $\omega_\nu = \xi_\nu + i\eta_\nu$ of $\hat{k}(\xi+i\eta)=1$ in $|\eta| \leqq a$, since $\hat{k}(\xi+i\eta) \to 0$ uniformly in this strip when $\xi \to \pm \infty$.

Note that by (1.11) any admissible solution G of (1.13) must satisfy (1.14). Further, since G must be real and positive we need consider only the zeros ω_ν which are purely imaginary, that is the $\omega_\nu = i\eta_\nu$ such that

$$\sin \pi\eta_\nu/\pi\eta_\nu = \alpha \qquad (-1 < \eta_\nu < 1). \tag{1.16}$$

It follows from (1.16) that $\alpha \leqq 1$. If $\alpha < 1$, there are precisely two values η_ν for which (1.16) is true, call them λ and $-\lambda$ $(0<\lambda<1)$. Then (1.15) reduces to

$$G(y) = Ae^{\lambda y} + Be^{-\lambda y}, \qquad g(x) = Ax^\lambda + Bx^{-\lambda}.$$

Since g is increasing and positive for $x>0$, necessarily $A>0$ and $B=0$, which proves (1.10) when $\alpha<1$.

If $\alpha=1$, then a glance at (1.16) and (1.15) shows that in this case

$$g(x) = A + B \log x,$$

with $A>0$ and $B=0$. This completes the proof of (1.10), and hence also of Theorem B.

2. **Further remarks on Theorem B.** One additional feature of the above proof should be explicitly pointed out: with only a single notational change (replace "$t_n \to \infty$" in the second paragraph of §1 by "$t_n \to 0$"), it also yields the following complement which describes the asymptotic behavior of the Stieltjes transform at the origin.

THEOREM C. *The statement of Theorem* B *remains true if* ∞ *is replaced by* 0 *in* (5), (2), (3) *and* (K).

Thus the existence of

$$\alpha = \lim_{t\to 0} \frac{\psi(t)}{tF(t)} \qquad (\alpha > 0) \tag{2.1}$$

implies that both ψ and F vary regularly at the origin. The converse statement, that if either ψ or F varies regularly at the origin then so does the other (and hence the limit in (2.1) exists), is due to Hardy and Littlewood [6].[3]

The motivation for a "ratio-tauberian" theorem for the Stieltjes transform arose from a problem in the value-distribution theory of entire functions, where the asymptotic behavior of ratios such as $N(r, 0)/\log M(r, f)$, as $r \to \infty$, is of key interest. Here, as in the usual notation of the theory,

$$M(r, f) = \max_\theta |f(re^{i\theta})|, \qquad N(r, 0) = \int_0^r \frac{n(t, 0)}{t}\, dt \quad (\text{if } f(0) \neq 0),$$

and $n(t, 0)$ denotes the number of zeros of the entire function f in the disk $|z| \leqq t$.

An important role is played in this theory by functions f of the form

$$f(z) = \prod_{\nu=1}^{\infty} \left(1 + \frac{z}{a_\nu}\right) \qquad (0 < a_\nu \leqq a_{\nu+1}), \tag{2.2}$$

since functions with negative zeros are "extremal" for many problems. Of particular interest are the "Lindelöf functions," i.e. functions of the form (2.2) with zeros $-a_\nu$ distributed so regularly that

$$N(r, 0) \sim r^\lambda L(r) \qquad (0 \leqq \lambda < 1, r \to \infty), \tag{2.3}$$

and Valiron's Theorem A asserts that (2.3) is equivalent to

$$\log M(r, f) \sim (\pi\lambda/\sin \pi\lambda)\, r^\lambda L(r) \qquad (r \to \infty). \tag{2.4}$$

(To see this, just notice that (2.2) implies

$$\log M(r, f) = \int_0^\infty \log\left(1 + \frac{r}{t}\right) dn(t, 0) = r \int_0^\infty \frac{dN(t, 0)}{r + t} \cdot \Big)$$

[3] Actually, the relevant result in [6, Theorem 4] is stated only for the case $L(t) \equiv$ constant, but the extension to general functions L varying slowly at the origin is an easy exercise.

Thus the Lindelöf functions satisfy

$$\lim_{r\to\infty} \frac{N(r, 0)}{\log M(r,f)} = \frac{\sin \pi\lambda}{\pi\lambda} \tag{2.5}$$

and the content of Theorem B is that the *only* functions of the form (2.2) which satisfy (2.5) are the Lindelöf functions.

Finally, we mention a generalization of Theorem B which—as D. Drasin has remarked—follows immediately from the above proof, and which will be convenient to have available for use in a subsequent paper.

THEOREM B′. *Let ψ and F be defined as in Theorem* A, *and let $G \subset (0, \infty)$ be any set of the form*

$$G = \bigcup_{n=1}^{\infty} (a_n, b_n) \qquad (a_n \to \infty,\ b_n/a_n \to \infty). \tag{2.6}$$

If the limit

$$\alpha = \lim_{t\to\infty;\, t\in G} \frac{\psi(t)}{tF(t)}$$

exists and is positive, then $\alpha \leqq 1$ and, if λ is determined by (6),

$$\lim_{n\to\infty} \frac{\psi(\sigma t_n)}{\psi(t_n)} = \sigma^{\lambda} \qquad (0 < \sigma < \infty) \tag{2.7}$$

holds for every sequence $\{t_n\}$ satisfying the condition

$$\sigma t_n \in G \qquad (0 < \sigma < \infty,\ n > n_0(\sigma)). \tag{2.8}$$

"Locally tauberian" theorems, in which conclusions like (2.7) are deduced from hypotheses somewhat different from those in the above statement, have recently been established by Edrei in his interesting paper [2].

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