A NOTE ON "NTH ROOTS OF OPERATORS"

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M. R. Embry has proved the following theorem [1]:

**Theorem [E].** "Let $A$ and $E$ be bounded operators on a Banach space, and suppose $\sigma(A) \cap \sigma(e^{2\pi i k/n}A) = \emptyset$ for $k = 1, 2, \ldots, n - 1$. Then if $E$ commutes with $A^n$, it commutes with $A$.

This result can be obtained as a corollary to the following theorem:

**Notation (as in [2]).** Let $T$ be a bounded operator on a Banach space. Let $\mathcal{F}(T) =$ the functions holomorphic on some neighborhood of $\sigma(T)$. Let $f \in \mathcal{F}(T)$.

As usual,

$$f(T) = (2\pi i)^{-1} \int_{\mathcal{C}} f(\lambda)(\lambda - T)^{-1}d\lambda.$$

**Theorem.** In order that $T$ be representable as a function of $f(T)$ it is sufficient that (1) $f$ is 1-1 on $\sigma(T)$, and (2) $f' \neq 0$ on $\sigma(T)$. Consequently if $E$ is a bounded operator and commutes with $f(T)$ then it commutes with $T$.

**Proof.** (1) and (2) ensure that $f$ is 1-1 on a neighborhood of $\sigma(T)$, for if not there exist $\{z_n\}, \{z'_n\}, z_n \neq z'_n$, such that $f(z_n) = f(z'_n)$ and these may be chosen so that $z_n \rightarrow z_0, z'_n \rightarrow z'_0, z_0 \in \sigma(T), z'_0 \in \sigma(T)$. Now $z_0 = z'_0$, else (1) is violated. But then

$$0 = \frac{f(z_n) - f(z'_n)}{z_n - z'_n} \rightarrow f'(z_0)$$

which contradicts (2). If $f$ is 1-1 on a neighborhood of $\sigma(T)$, then $f^{-1} \in \mathcal{F}(f(T))$, and Theorem 12 [2, p. 570] yields $T = f^{-1}(f(T))$.

If $f$ fails to satisfy (1) or (2), an $E$ and $T$ for which $E$ commutes with $f(T)$ but not with $T$ can be found in $2 \times 2$ matrices.

Note that $z^n$ satisfies (1) and (2) precisely if the condition in Theorem E is satisfied.

**References**


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250