A NOTE ON COMPLETE SEPARATION IN THE STONE TOPOLOGY

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Let L be the lattice of all ideals of a ring with unity and let X be a subset of the set of all real ideals of A [**1**] which is equipped with the Stone topology [**4**, p. 272]. Anderson [**1**, Lemma 4, 5(2)] gave necessary and sufficient conditions, in terms of certain elements of L, for two subsets of X to be completely separated in X. In [**6**, §8], we modified the idea of Anderson in case L is the lattice of all S_α-ideals of a C-lattice [**2**, §§2 and 3] for α a real number and X a certain subset of L equipped with the Stone topology. The purpose of this note is to give necessary and sufficient conditions for two subsets of X to be completely separated in X in case L is an arbitrary complete lattice and X is a subset of L which is completely regular when equipped with the Stone topology. The idea is essentially that of Anderson in [**1**]. We use this result to complete the "internal" characterization of the Φ-algebra of all real-valued continuous functions on an arbitrary completely regular space for which Henriksen and Johnson [**6**] have results in special cases.

Suppose now that L is a complete lattice and X is a fixed subset of L. We define a function ϕ from L into L by

$$\phi(x) = \bigwedge\{y \in X \mid x \leqq y\} \qquad (x \in L)$$

and we let $K = \{x \in L \mid x = \phi(x)\}$. It is clear that $\phi(x)$ is merely the meet of the intersection of X with the principal dual ideal of L generated by x. It is also a routine matter to verify that ϕ is a closure operation [**3**, p. 49] on L and hence by [**8**, Theorem 4.1], K is a complete lattice relative to the partial order on L. Moreover, meets in K coincide with meets in L and joins in K are obtained by operating on the joins in L by ϕ. In addition, $X \subset K$, $\bigwedge X = \bigwedge K$, and if $1 = \bigvee L$, then $1 \in K$.

We now suppose that X is also equipped with the Stone topology. Using the fact that the closure of a subset Y of X is $\{y \in X \mid \bigwedge Y \leqq y\}$, one can show that the mapping $F \to \bigwedge F$ is a dual isomorphism of the lattice of all closed subsets of X onto K.

Next, we define a binary relation $*$ on K as follows: $x * y$ in case

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there exists $z \in K$ such that $z \wedge x = \bigwedge X$ and $z \vee y = 1$. The motivation for this definition is the following: If F_1 and F_2 are closed subsets of X, then $F_2 \subset \operatorname{int} F_1$ if and only if $\bigwedge F_1 * \bigwedge F_2$. Note that this implies that $*$ is a transitive relation on K.

If $C \subset K$, we say that C is a *dense chain with respect to* $*$ in Case (i) for each $x, y \in C$, either $x * y$ or $y * x$, and (ii) if $x, y \in C$ with $x * y$, then there exists $z \in C$ such that $x * z * y$. We now define a binary relation $\perp$ on K by $x \perp y$ in case there exists a countable subset C of K such that C is a dense chain with respect to $*$, $x \leqq \bigwedge C$, and $\bigvee C \leqq y$.

THEOREM 1. *Let X be a subset of a complete lattice L so that X is a completely regular space when equipped with the Stone topology. Subsets A_1 and A_2 of X are completely separated in X if and only if there exist closed subsets F_1 and F_2 of X such that $A_1 \subset X \backslash F_1$, $A_2 \subset F_2$, and $\bigwedge F_1 \perp \bigwedge F_2$.*

PROOF. Let Q denote the rational numbers. It is not difficult to show that subsets A_1 and A_2 of a completely regular space X are completely separated in X if and only if there exist closed sets F_1 and F_2 of X and a family $(U_t)_{t \in Q}$ of open sets of X such that $A_1 \subset X \backslash F_1 \subset \bigcap_{t \in Q} U_t$, $\bigcup_{t \in Q} U_t \subset X \backslash F_2 \subset X \backslash A_2$, and $r < s$ in Q implies that $\operatorname{cl} U_r \subset U_s$.

Suppose first that A_1 and A_2 are completely separated in X, and let F_1, F_2, and $(U_t)_{t \in Q}$ be as described above. For each $t \in Q$, let $F_t = X \backslash U_t$. One can show that $\{\bigwedge F_t \mid t \in Q\}$ is a countable dense chain with respect to $*$ such that $\bigwedge F_1 \leqq \bigwedge \{\bigwedge F_t \mid t \in Q\}$ and $\bigvee \{\bigwedge F_t \mid t \in Q\} \leqq \bigwedge F_2$.

Conversely, if there exist closed sets F_1 and F_2 of X such that $A_1 \subset X \backslash F_1$, $A_2 \subset F_2$, and $\bigwedge F_1 \perp \bigwedge F_2$, then there exists a countable subset C of K such that C is a dense chain with respect to $*$, $\bigwedge F_1 \leqq \bigwedge C$, and $\bigvee C \leqq \bigwedge F_2$. The conditions on C imply that there exists a function $f: Q \rightarrow C$ such that $r < s$ in Q implies that $f(r) * f(s)$. For each $t \in Q$, let U_t be the open subset of X defined by $\bigwedge (X \backslash U_t) = f(t)$. It is a routine matter to verify that $(U_t)_{t \in Q}$ is a family of open sets satisfying $X \backslash F_1 \subset \bigcap_{t \in Q}$, $\bigcup_{t \in Q} U_t \subset X \backslash F_2$, and $r < s$ in Q implies that $\operatorname{cl} U_r \subset U_s$. Hence the theorem.

Suppose now that A is a Φ-algebra. We refer the reader to [5] for definitions, symbols, and notation not defined here. Let L be the complete lattice of all l-ideals of A, let $\mathcal{R}(A)$ be the set of all real maximal l-ideals of A equipped with the Stone topology, and let $K(A)$ be the subset of L as defined earlier using the operator ϕ and the subset $\mathcal{R}(A)$ of L. It follows that $I \in K(A)$ if and only if

$I=\bigcap\{M\in\mathcal{R}(A)\mid I\subset M\}$. If $\bigcap\mathcal{R}(A)=\{0\}$, the binary relation $*$ can be restated slightly as follows: $I_1 * I_2$ in case there exists $J\in K$ with $I_1\cap J=\{0\}$ and $I_2\cup J\not\subset M$ for any $M\in\mathcal{R}(A)$.

We can now obtain an "internal" characterization as a Φ-algebra of the set $C(X)$ of all real-valued continuous functions on an arbitrary completely regular space X.

THEOREM 2 (COMPARE [5, 5.2]). *A Φ-algebra A is isomorphic to $C(X)$ for some completely regular space X if and only if*

(i) *A is an algebra of real-valued functions,*

(ii) *A is uniformly closed,*

(iii) *A is closed under inversion, and*

(iv) *for each pair $I\perp J$ in $K(A)$, there exists $f\in A$ such that $f-1\in J$ and $f\wedge|h|=0$ for all $h\in I$.*

PROOF. The only condition which needs comment concerning the necessity of the conditions is (iv). It follows easily from 4.6 in [5] and the fact that $I\perp J$ in $K(A)$ if and only if the corresponding sets in $\mathcal{R}(A)$ are completely separated.

In view of the proof of 5.2 in [5], the sufficiency of the four conditions will be immediate if it is shown that disjoint zero-sets in $\mathcal{R}(A)$ have disjoint closures in $\mathfrak{M}(A)$, where $\mathfrak{M}(A)$ is the space of all maximal l-ideals of A. Let Z_1 and Z_2 be disjoint zero-sets in $\mathcal{R}(A)$. Then there exist closed subsets F_1 and F_2 in $\mathcal{R}(A)$ such that $Z_1\subset\mathcal{R}(A)\setminus F_1$, $Z_2\subset F_2$, and $\bigcap F_1\perp\bigcap F_2$. By (iv), there exists $f\in A$ such that $f-1\in\bigcap F_2$ and $f\wedge|h|=0$ for all $h\in\bigcap F_1$. Thus, if $M\in Z_1$, then $M\notin F_1$ and so $\bigcap F_1\not\subset M$. Hence there exists $h\in\bigcap F_1$ such that $h\notin M$. Since M is an l-ideal, $|h|\notin M$. By [7, I, 3.15 (iii)], $f\wedge|h|=0$ implies that $f|h|=0$ and so $f|h|\in M$. By the primeness of M, $f\in M$ and thus $\bar{f}(M)=0$. It is easy to see that if $M\in Z_2$, then $\bar{f}(M)=1$. Moreover, we may assume that $0\leqq\bar{f}(M)\leqq 1$ for all $M\in\mathcal{R}(A)$ since $(f\vee 0)\wedge 1\in A$. Thus $\bar{f}(M)\in R$ for all $M\in\mathfrak{M}(A)$ and so $\{M\in\mathfrak{M}(A)\mid\bar{f}(M)=0\}$ and $\{M\in\mathfrak{M}(A)\mid\bar{f}(M)=1\}$ are disjoint closed sets in $\mathfrak{M}(A)$ containing Z_1 and Z_2, respectively.

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