REAL REPRESENTATIONS OF METACYCLIC GROUPS

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Problem 14 in R. Brauer's survey article *Representations of finite groups* [1] asks for a characterization of the number of irreducible complex representations of a group G that are equivalent to representations over the real field $\mathbf{R}$. The question is answered here for a class of metacyclic groups, the answer being an arithmetic function of the parameters that appear in presentations of the groups.

Suppose G is a metacyclic group having a cyclic normal subgroup $A=\langle a\rangle$ of order m and cyclic quotient group $G/A=\langle bA\rangle$ of order s, with $b^{-1}ab=a^r$ and $b^s=a^t$, $1\leqq r\leqq m-1$, $0\leqq t\leqq m-1$. Then by [2, p. 334], we have

$$(m, r) = 1, \qquad tr \equiv t(\bmod m), \quad \text{and} \quad r^s \equiv 1(\bmod m).$$

For later use we set $d=(m, t, r^{s/2}+1)$ when s is even.

If $\zeta\in\mathbf{C}$ is a primitive mth root of unity then all irreducible complex characters of A are obtained by setting $\phi_i(a)=\zeta^i$, $i=0, \cdots, m-1$. Each ϕ_i gives rise to an induced character $\theta_i=\phi_i{}^G$ of G, and we wish to investigate which of the irreducible characters among the θ_i are characters of representations over $\mathbf{R}$. The induced character θ_i is defined by the formula

$$\theta_i(x) = (1/m)\sum_{t\in G}\phi_i(t^{-1}xt) \qquad \text{if } x\in A,$$
$$= 0 \qquad \text{if } x\notin A.$$

Let us compute θ_i more explicitly. We have in general $b^{-j}ab^j=a^{r^j}$, and so

$$\theta_i(a^n) = (1/m)\sum_{j=0}^{s-1}\sum_{k=0}^{m-1}\phi_i(a^{-k}b^{-j}a^nb^ja^k)$$
$$= \sum_{j=0}^{s-1}\phi_i(b^{-j}ab^j)^n = \sum_{j=0}^{s-1}\zeta^{ir^jn}.$$

For each θ_i set $\nu(\theta_i)=(1/ms)\sum_{x\in G}\theta_i(x^2)$. If θ_i is irreducible then $\nu(\theta_i)=1$ if and only if θ_i is the character of a real representation, by the Theorem of Frobenius and Schur [3, p. 22].

PROPOSITION 1. *Suppose s is odd. Then $\nu(\theta_i)=0$ for every irreducible θ_i, and so no irreducible θ_i is the character of a real representation.*

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PROOF. We have $(b^k a^n)^2 = b^{2k} a^{n(r^k+1)}$, so $(b^k a^n)^2 \in A$ if and only if $s \mid k$, since s is odd. We may assume $0 \leqq k \leqq s-1$ and still obtain all elements of G in the form $b^k a^n$, so $(b^k a^n)^2 \in A$ if and only if $k=0$. Thus

$$\nu(\theta_i) = (1/ms) \sum_{k=0}^{s-1} \sum_{n=0}^{m-1} \theta_i(b^{2k} a^{n(r^k+1)})$$

$$= (1/ms) \sum_{n=0}^{m-1} \theta_i(a^{2n}) = (1/ms) \sum_{n=0}^{m-1} \sum_{j=0}^{s-1} \zeta^{2inr}$$

$$= (1/ms) \sum_j \sum_n (\zeta^{2ir^j})^n.$$

If $m \nmid 2i$, then $m \nmid 2ir^j$ since $(m, r)=1$, and then $\sum_n (\zeta^{2ir^j})^n = (1-\zeta^{2ir^j m})/(1-\zeta^{2ir^j}) = 0$, and so $\nu(\theta_i)=0$. If $m \mid 2i$ then either $i=0$ or $i=m/2$, with m even. In either case each summand is 1, so $\nu(\theta_i)=1$. Observe, however, that if m is even then $r-1$ is even, and we have $m \mid (m/2)(r-1)$. Thus θ_i is reducible if $i=m/2$ (see [2, p. 335]). Likewise θ_i is reducible if $i=0$, and the proof is complete.

In view of Proposition 1 *we suppose from this point on that s is even.*

THEOREM 1. *If θ_i is irreducible, then $\nu(\theta_i)=1$ if and only if $i \equiv 0 \pmod{m/d}$.*

PROOF. We have $\theta_i((b^k a^n)^2) = \theta_i(b^{2k} a^{n(r^k+1)}) = 0$ unless $k=0$ or $k=s/2$. Thus

$$\nu(\theta_i) = (1/ms) \sum_{n=0}^{m-1} [\theta_i(a^{2n}) + \theta_i(a^{t+n(r^{s/2}+1)})]$$

$$= (1/ms) \sum_n \left[\sum_{j=0}^{s-1} \zeta^{2inr^j} + \sum_{j=0}^{s-1} \zeta^{inr^j(r^{s/2}+1)+itr^j} \right].$$

Since $tr \equiv t \pmod m$ we have $tr^j \equiv t \pmod m$, and so $\zeta^{itr^j} = \zeta^{it}$. Observe that

$$\sum_{n=0}^{m-1} (\zeta^{2ir^j})^n = m \qquad \text{if } m \mid 2i,$$
$$= 0 \qquad \text{if } m \nmid 2i,$$

and that

$$\sum_{n=0}^{m-1} \zeta^{it}(\zeta^{ir^j(r^{s/2}+1)})^n = m\zeta^{it} \qquad \text{if } m \mid i(r^{s/2}+1),$$
$$= 0 \qquad \text{if } m \nmid i(r^{s/2}+1).$$

Thus $\nu(\theta_i) = B + D\zeta^{it}$, where

$$B = 1 \quad \text{if } m \mid 2i, \qquad D = 1 \quad \text{if } m \mid i(r^{s/2}+1),$$
$$\quad = 0 \quad \text{if } m \nmid 2i, \qquad \quad = 0 \quad \text{if } m \nmid i(r^{s/2}+1).$$

But $m \mid 2i$ only if $i = m/2$, with m even, and in that case θ_i is reducible, as shown above. Thus $B=0$ and $\nu(\theta_i)=1$ if and only if $D=\zeta^{it}=1$, i.e. if and only if $m \mid i(r^{s/2}+1)$ and $m \mid it$.

In other words, we want all common solutions to the pair of congruences $i(r^{s/2}+1)\equiv 0 \pmod m$, $it\equiv 0 \pmod m$ in the range $1 \leqq i \leqq m-1$. Since $it \equiv 0 \pmod m$ if and only if $i\equiv 0(\text{mod } m/(m, t))$, and $i(r^{s/2}+1)\equiv 0 \pmod m$ if and only if $i \equiv 0(\text{mod } m/(m, r^{s/2}+1))$, we have a common solution if and only if

$$i \equiv 0(\text{mod}[m/(m, t), m/(m, r^{s/2}+1)]).$$

But $[m/(m, t), m/(m, r^{s/2}+1)] = m/d$, so we have shown that $\nu(\theta_i)=1$ if and only if $i \equiv 0(\text{mod } m/d)$.

Let us restate the original question in the light of Theorem 1. For each k, $1 \leqq k \leqq d-1$, set $\psi_k = \theta_{km/d}$. We wish to determine which of the characters ψ_k are irreducible and which ones are equal to one another, and then to count the distinct irreducible ψ_k.

Proposition 2. *The character ψ_k is reducible if and only if $kr^j \equiv k \pmod d$ for some j, $1 \leqq j \leqq s-1$.*

Proof. It is shown in [2, p. 335] that ψ_k is reducible if and only if $m \mid km(r^j-1)/d$ for some such j, and that is equivalent with the stated proposition.

Similarly, we have

Proposition 3. *$\psi_k = \psi_n$ if and only if $kr^j \equiv n \pmod d$ for some j, $1 \leqq j \leqq s-1$.*

Those metacyclic groups all of whose irreducible characters are either one dimensional or else among the characters θ_i are characterized in [2, p. 336]. We are now in a position to answer Brauer's question for such groups.

Theorem 2. *Suppose that G is metacyclic and that all nonlinear irreducible characters of G are induced from the cyclic subgroup A. If s is even then G has exactly $(d-(d, r-1))/s$ inequivalent absolutely irreducible nonlinear representations over the real field $\mathbf{R}$. If s is odd, then G has no absolutely irreducible nonlinear representations over $\mathbf{R}$.*

Proof. The case where s is odd is covered by Proposition 1. Suppose then that s is even. By [2, p. 336] we know that if $r^j i \equiv i \pmod m$, with $1 \leqq j \leqq s-1$, then $ri \equiv i (\mathrm{mod}\ m)$. Set $Y = \{1, 2, \cdots, d\}$, and define $\beta: Y \rightarrow Y$ by setting $\beta(k) = rk (\mathrm{mod}\ d)$. Since $(r, d) = 1$, β is a permutation of Y. Since $r^s \equiv 1 (\mathrm{mod}\ d)$ the order of β divides s. Note that if $\beta^j(k) = r^j k = k$ in the integers mod d, then in fact $\beta(k) = k$, if $j \leqq s-1$. It follows that the orbits of β in Y have either one or s elements. It is clear from Propositions 2 and 3 that ψ_k is reducible if and only if $\{k\}$ is a one-element orbit, and that $\psi_k = \psi_n$ if and only if k and n are in the same orbit. But $\{k\}$ is an orbit if and only if $d | k(r-1)$, i.e. if and only if

$$k = d/(d, r - 1), 2d/(d, r - 1), \cdots, d = (d, r - 1)d/(d, r - 1).$$

Thus the number of s-element orbits, and hence the number of ψ_k that are characters of real representations, is $(d - (d, r-1))/s$.

REMARK. Note that $r^n + 1 = (r-1) \sum_{k=1}^{n-1} r^k + (r+1)$. Thus for $n \geqq 1$ we have

$$\begin{aligned} (r - 1, r^n + 1) = (r - 1, r + 1) &= 1 \qquad \text{if } r \text{ is even}, \\ &= 2 \qquad \text{if } r \text{ is odd}. \end{aligned}$$

As a result $(d, r-1) = (m, t, r+1, r-1) = 2$ if r is odd and m and t are both even, and $(d, r-1) = 1$ in all other cases.

Let us consider Theorem 2 for two particular examples.

1. The dihedral group D_m of order $2m$ is metacyclic, with $s = 2$, $t = 0$, and $r = m-1$. Thus $d = (m, m) = m$, and $(d, r-1) = 2$ if m is even, $= 1$ if m is odd. We conclude that D_m has $(m-2)/2$ nonlinear absolutely irreducible real representations if m is even, and $(m-1)/2$ if m is odd.

2. The generalized quaternion group Q_n of order $4n$ is metacyclic, with $m = 2n$, $s = 2$, $t = n$, and $r = 2n-1$. Then $d = (2n, n, 2n) = n$, and $(d, r-1) = 1$ if n is odd, $= 2$ if n is even. Thus Q_n has $(n-1)/2$ nonlinear absolutely irreducible real representations if n is odd, and $(n-2)/2$ if n is even.

COROLLARY TO THEOREM 2. *Suppose G is a finite nonabelian group having a cyclic subgroup of index 2. Then all nonlinear irreducible complex representations of G are equivalent with real representations if and only if G is a dihedral group D_m.*

The proof will be omitted. The referee has pointed out that the conclusion of the corollary holds under the hypotheses that $[G: \langle a \rangle] = 2$ and G has an absolutely irreducible faithful nonlinear representation over $\mathbf{R}$.

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