REAL REPRESENTATIONS OF METACYCLIC GROUPS

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Problem 14 in R. Brauer's survey article Representations of finite groups [1] asks for a characterization of the number of irreducible complex representations of a group $G$ that are equivalent to representations over the real field $\mathbb{R}$. The question is answered here for a class of metacyclic groups, the answer being an arithmetic function of the parameters that appear in presentations of the groups.

Suppose $G$ is a metacyclic group having a cyclic normal subgroup $A = \langle a \rangle$ of order $m$ and cyclic quotient group $G/A = \langle bA \rangle$ of order $s$, with $b^{-1}ab = a^r$ and $b^t = a^t$, $1 \leq r \leq m-1$, $0 \leq t \leq m-1$. Then by [2, p. 334], we have

$$(m, r) = 1, \quad tr \equiv t(\text{mod } m), \quad \text{and } \quad r^* = 1(\text{mod } m).$$

For later use we set $d = (m, t, r^{s+1})$ when $s$ is even.

If $\xi \in \mathbb{C}$ is a primitive $m$th root of unity then all irreducible complex characters of $A$ are obtained by setting $\phi_i(a) = \xi^i$, $i = 0, \ldots, m-1$. Each $\phi_i$ gives rise to an induced character $\theta_i = \phi_i^G$ of $G$, and we wish to investigate which of the irreducible characters among the $\theta_i$ are characters of representations over $\mathbb{R}$. The induced character $\theta_i$ is defined by the formula

$$\theta_i(x) = \left(\frac{1}{m}\right) \sum_{t \in \mathbb{Z}} \phi_i(t^{-1}xt) \quad \text{if } x \in A,$$

$$= 0 \quad \text{if } x \notin A.$$

Let us compute $\theta_i$ more explicitly. We have in general $b^{-i}ab^i = a^t$, and so

$$\theta_i(a^n) = \left(\frac{1}{m}\right) \sum_{j=0}^{s-1} \sum_{k=0}^{m-1} \phi_i(a^{-k}b^{-i}a^nb^{ik})$$

$$= \sum_{j=0}^{s-1} \phi_i(b^{-i}ab^i)^n = \sum_{j=0}^{s-1} \xi^{tn}.$$

For each $\theta_i$ set $\nu(\theta_i) = (1/ms) \sum_{x \in G} \theta_i(x^2)$. If $\theta_i$ is irreducible then $\nu(\theta_i) = 1$ if and only if $\theta_i$ is the character of a real representation, by the Theorem of Frobenius and Schur [3, p. 22].

PROPOSITION 1. Suppose $s$ is odd. Then $\nu(\theta_i) = 0$ for every irreducible $\theta_i$, and so no irreducible $\theta_i$ is the character of a real representation.

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Proof. We have \((b^ka^n)^2 = b^{2k}a^{n(r^k + 1)}\), so \((b^ka^n)^2 \in A\) if and only if \(s \mid k\), since \(s\) is odd. We may assume \(0 \leq k \leq s-1\) and still obtain all elements of \(G\) in the form \(b^ka^n\), so \((b^ka^n)^2 \in A\) if and only if \(k = 0\). Thus

\[
\nu(\theta_i) = (1/ms) \sum_{k=0}^{s-1} \sum_{n=0}^{m-1} \theta_i(b^{2k}a^{n(r^k + 1)})
\]

\[
= (1/ms) \sum_{n=0}^{m-1} \theta_i(a^{2n}) = (1/ms) \sum_{n=0}^{m-1} \sum_{j=0}^{s-1} \xi^{2nirr}
\]

\[
= (1/ms) \sum_j \sum_n (\xi^{2ir})^n.
\]

If \(m \mid 2i\), then \(m \mid 2ir\) since \((m, r) = 1\), and then \(\sum_n (\xi^{2ir})^n = (1 - \xi^{2irm})/(1 - \xi^{2ir}) = 0\), and so \(\nu(\theta_i) = 0\). If \(m \nmid 2i\), then either \(i = 0\) or \(i = m/2\), with \(m\) even. In either case each summand is 1, so \(\nu(\theta_i) = 1\). Observe, however, that if \(m\) is even then \(r - 1\) is even, and we have \(m \mid (m/2)(r - 1)\). Thus \(\theta_i\) is reducible if \(i = m/2\) (see [2, p. 335]). Likewise \(\theta_i\) is reducible if \(i = 0\), and the proof is complete.

In view of Proposition 1 we suppose from this point on that \(s\) is even.

Theorem 1. If \(\theta_i\) is irreducible, then \(\nu(\theta_i) = 1\) if and only if \(i \equiv 0 (mod m/d)\).

Proof. We have \(\theta_i((b^ka^n)^2) = \theta_i(b^{2k}a^{n(r^k + 1)}) = 0\) unless \(k = 0\) or \(k = s/2\). Thus

\[
\nu(\theta_i) = (1/ms) \sum_{n=0}^{m-1} \left[ \theta_i(a^{2n}) + \theta_i(a^{s+n(r^{s/2} + 1)}) \right]
\]

\[
= (1/ms) \sum_n \left[ \sum_{j=0}^{s-1} \xi^{2nir} + \sum_{j=0}^{s-1} \xi^{nir(r^{s/2} + 1) + irr} \right].
\]

Since \(tr \equiv t (mod m)\) we have \(tr' \equiv t (mod m)\), and so \(\xi^ir' = \xi^i\). Observe that

\[
\sum_{n=0}^{m-1} (\xi^{2ir})^n = m \quad \text{if } m \mid 2i,
\]

\[
= 0 \quad \text{if } m \nmid 2i,
\]

and that

\[
\sum_{n=0}^{m-1} \xi^i(\xi^{ir'(r^{s/2} + 1)})^n = m\xi^i \quad \text{if } m \mid i(r^{s/2} + 1),
\]

\[
= 0 \quad \text{if } m \nmid i(r^{s/2} + 1).
\]

Thus \(\nu(\theta_i) = B + D\xi^i\), where
But \( m \mid 2i \) only if \( i = m/2 \), with \( m \) even, and in that case \( \theta_i \) is reducible, as shown above. Thus \( B = 0 \) and \( \nu(\theta_i) = 1 \) if and only if \( D = \xi^{it} = 1 \), i.e. if and only if \( m \mid i(r^{it}/2 + 1) \) and \( m \mid it \).

In other words, we want all common solutions to the pair of congruences \( i(r^{it}/2 + 1) \equiv 0 \pmod{m} \), \( it \equiv 0 \pmod{m} \) in the range \( 1 \leq i \leq m - 1 \). Since \( it \equiv 0 \pmod{m} \) if and only if \( i \equiv 0 \pmod{m/(m, t)} \), and \( i(r^{it}/2 + 1) \equiv 0 \pmod{m} \) if and only if \( i \equiv 0 \pmod{m/(m, r^{it}/2 + 1)} \), we have a common solution if and only if

\[
i \equiv 0 \pmod{\left[ m/(m, t), m/(m, r^{it}/2 + 1) \right]}.
\]

But \( \left[ m/(m, t), m/(m, r^{it}/2 + 1) \right] = m/d \), so we have shown that \( \nu(\theta_i) = 1 \) if and only if \( i \equiv 0 \pmod{m/d} \).

Let us restate the original question in the light of Theorem 1. For each \( k, 1 \leq k \leq d - 1 \), set \( \psi_k = \theta_{km/d} \). We wish to determine which of the characters \( \psi_k \) are irreducible and which ones are equal to one another, and then to count the distinct irreducible \( \psi_k \).

**Proposition 2.** The character \( \psi_k \) is reducible if and only if \( kr^i \equiv k \pmod{d} \) for some \( j, 1 \leq j \leq s - 1 \).

**Proof.** It is shown in [2, p. 335] that \( \psi_k \) is reducible if and only if \( m \mid km(r^i - 1)/d \) for some such \( j \), and that is equivalent with the stated proposition.

Similarly, we have

**Proposition 3.** \( \psi_k = \psi_n \) if and only if \( kr^i \equiv n \pmod{d} \) for some \( j, 1 \leq j \leq s - 1 \).

Those metacyclic groups all of whose irreducible characters are either one dimensional or else among the characters \( \theta_i \) are characterized in [2, p. 336]. We are now in a position to answer Brauer's question for such groups.

**Theorem 2.** Suppose that \( G \) is metacyclic and that all nonlinear irreducible characters of \( G \) are induced from the cyclic subgroup \( A \). If \( s \) is even then \( G \) has exactly \((d - (d, r - 1))/s\) inequivalent absolutely irreducible nonlinear representations over the real field \( \mathbb{R} \). If \( s \) is odd, then \( G \) has no absolutely irreducible nonlinear representations over \( \mathbb{R} \).

**Proof.** The case where \( s \) is odd is covered by Proposition 1. Suppose then that \( s \) is even. By [2, p. 336] we know that if \( r^i \equiv i \pmod{m} \),
with \(1 \leq j \leq s-1\), then \(ri \equiv i \pmod{m}\). Set \(Y = \{1, 2, \ldots, d\}\), and define \(\beta: Y \to Y\) by setting \(\beta(k) = r^j k \pmod{d}\). Since \((r, d) = 1\), \(\beta\) is a permutation of \(Y\). Since \(r^s = 1 \pmod{d}\) the order of \(\beta\) divides \(s\). Note that if \(\beta'(k) = r^j k = k\) in the integers mod \(d\), then in fact \(\beta(k) = k\), if \(j \leq s-1\). It follows that the orbits of \(\beta\) in \(Y\) have either one or \(s\) elements. It is clear from Propositions 2 and 3 that \(\psi_k\) is reducible if and only if \(\{k\}\) is a one-element orbit, and that \(\psi_k = \psi_{k'}\) if and only if \(k\) and \(k'\) are in the same orbit. But \(\{k\}\) is an orbit if and only if \(d \mid k(r-1)\), i.e. if and only if

\[
  k = \frac{d}{(d, r - 1)}, \frac{2d}{(d, r - 1)}, \ldots, d = (d, r - 1)\frac{d}{(d, r - 1)}.
\]

Thus the number of \(s\)-element orbits, and hence the number of \(\psi_k\) that are characters of real representations, is \((d - (d, r - 1))/s\).

**Remark.** Note that \(r^n + 1 = (r - 1) \sum_{i=0}^{n-1} r^i + (r+1)\). Thus for \(n \geq 1\) we have

\[
  (r - 1, r^n + 1) = (r - 1, r + 1) = 1 \quad \text{if } r \text{ is even},
\]

\[
  = 2 \quad \text{if } r \text{ is odd}.
\]

As a result \((d, r - 1) = (m, t, r + 1, r - 1) = 2\) if \(r\) is odd and \(m\) and \(t\) are both even, and \((d, r - 1) = 1\) in all other cases.

Let us consider Theorem 2 for two particular examples.

1. The dihedral group \(D_m\) of order \(2m\) is metacyclic, with \(s = 2\), \(t = 0\), and \(r = m - 1\). Thus \(d = (m, m) = m\), and \((d, r - 1) = 2\) if \(m\) is even, \(= 1\) if \(m\) is odd. We conclude that \(D_m\) has \((m - 2)/2\) nonlinear absolutely irreducible real representations if \(m\) is even, and \((m - 1)/2\) if \(m\) is odd.

2. The generalized quaternion group \(Q_n\) of order \(4n\) is metacyclic, with \(m = 2n\), \(s = 2\), \(t = n\), and \(r = 2n - 1\). Then \(d = (2n, n, 2n) = n\), and \((d, r - 1) = 1\) if \(n\) is odd, \(= 2\) if \(n\) is even. Thus \(Q_n\) has \((n - 1)/2\) nonlinear absolutely irreducible real representations if \(n\) is odd, and \((n - 2)/2\) if \(n\) is even.

**Corollary to Theorem 2.** Suppose \(G\) is a finite nonabelian group having a cyclic subgroup of index 2. Then all nonlinear irreducible complex representations of \(G\) are equivalent with real representations if and only if \(G\) is a dihedral group \(D_m\).

The proof will be omitted. The referee has pointed out that the conclusion of the corollary holds under the hypotheses that \([G: \langle a \rangle] = 2\) and \(G\) has an absolutely irreducible faithful nonlinear representation over \(R\).

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REFERENCES


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