

THE NILPOTENCY OF BANACH NIL ALGEBRAS

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THEOREM. *If B is a Banach algebra which is also a nil algebra, then B is a nilpotent algebra.*

PROOF. For each positive integer j , define

$$N_j = \{x \in B: x^j = 0\}.$$

By the theorem of Nagata-Higman [1, p. 274], it will be enough to show that some $N_j = B$. Each N_j is closed and the union of the N_j is B . Hence, by the Baire Category Theorem, there is a fixed integer k and a fixed $z \in B$ for which N_k is a neighborhood of z . Suppose $x \in B$. Define the B valued polynomial of a scalar variable t , $p(t) = (z + t(x - z))^k$. Since $z + t(x - z)$ is continuous, $p(t)$ must equal zero for all sufficiently small t . Therefore, $p(t) \equiv 0$; and, in particular $p(1) = x^k = 0$. This completes the proof.

It is easy to construct examples of normed algebras for which the above theorem is false, though the theorem is true for second category topological algebras.

REFERENCE

1. N. Jacobson, *Structure of rings*, rev. ed., Amer. Math. Soc. Colloq. Publ., Vol. 37, Amer. Math. Soc., Providence, R. I., 1964.

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