

ON THE JACOBIAN IDEAL OF THE MODULE OF DIFFERENTIALS¹

JOSEPH LIPMAN

Let A be a reduced algebra of finite type over a field k , and let J be the Jacobian ideal of A/k (i.e. J is the smallest nonzero Fitting ideal of the module of k -differentials of A). In some geometric investigations it has been found useful to consider the scheme V obtained by blowing up J . (Loosely speaking, the geometric points of V over a point Q of $\text{Spec}(A)$ correspond to the various "limiting positions" of the tangent spaces at points P approaching Q along arcs which are simple on $\text{Spec}(A)$.) It has even been proposed that one try to resolve singularities by successively blowing up Jacobian ideals. Implicit in this proposal is the assumption that *if J is invertible, then $\text{Spec}(A)$ is nonsingular*. Our purpose is to show, among other things, that this is indeed so when k has characteristic zero.

Before stating the theorem we recall some definitions. Let M be a (unitary) module of finite type over a ring S (commutative, with identity) generated, say, by e_1, e_2, \dots, e_n . Let (a_{ij}) (i in some indexing set; $j=1, 2, \dots, n$) be a matrix with entries in S , such that for each i , $\sum_{j=1}^n a_{ij}e_j=0$, and such that any row-vector (a_1, a_2, \dots, a_n) for which $\sum a_j e_j=0$, is a linear combination with coefficients in S of the rows of (a_{ij}) ; in other words the rows of (a_{ij}) generate the "module of relations" of e_1, e_2, \dots, e_n . For an integer $p \geq 0$, the $(n-p) \times (n-p)$ minors of (a_{ij}) generate an ideal $I_p(M)$; by convention $I_p(M)=S$ for $p \geq n$. The ideal $I_p(M)$ is known to depend neither on the choice of the basis $\{e_1, e_2, \dots, e_n\}$ nor on the choice of the "relation matrix" (a_{ij}) ; $I_p(M)$ is the p th *Fitting ideal* of M (cf. [2]). We have

$$I_0 \subseteq I_1 \subseteq \dots \subseteq I_n = I_{n+1} = \dots = S.$$

If I is the annihilator of M , then $I^n \subseteq I_0 \subseteq I$.

The *torsion submodule* M^t of M consists of all elements m in M such that m is annihilated by a *regular element* (i.e. a nonzerodivisor) in S . We denote by $\dim \text{proj}(M)$ the *projective dimension* of M , i.e. the smallest length (possibly infinite) of a projective resolution of M .

THEOREM. *Let k be a field and let R be a reduced local ring of the form*

Received by the editors July 25, 1968.

¹ Research supported by NSF-GP-6388 at Purdue University.

$A_{\mathfrak{p}}$, where A is a finitely generated k -algebra and \mathfrak{p} is a prime ideal in A . Assume that for each minimal prime \mathfrak{q} of R , the field $R_{\mathfrak{q}}$ is separable over k . Let $D = \Omega_{R/k}^1$ be the module of k -differentials of R , with torsion submodule D^t , and let J be the smallest nonzero Fitting ideal of D . Then the following conditions are equivalent:

(1) J is principal, generated by a regular element in R (i.e. J is invertible).

(2) R is a complete intersection, and D/D^t is free.

Moreover, if k has characteristic zero, then D/D^t is free if and only if R is regular.

PROOF. R being as above, Ferrand has shown (cf. [1]) that R is a complete intersection if and only if $\dim \text{proj}(D) \leq 1$. The equivalence of (1) and (2) results therefore from:

LEMMA 1. Let S be a local ring (not necessarily noetherian) and let M be a finitely generated S -module, with torsion submodule M^t . Let r be a nonnegative integer. The following conditions are equivalent:

(i) The smallest nonzero Fitting ideal of M is $I_r(M)$, and $I_r(M)$ is generated by a single regular element of S .

(ii) M is of finite presentation, $\dim \text{proj}(M) \leq 1$, and M/M^t is free of rank r .

PROOF. (ii) \Rightarrow (i). Assuming (ii) we have an isomorphism of S -modules $M \cong M^t \oplus S^r$ whence M^t is of finite presentation, and for $p \geq 0$

$$I_p(M) = I_{p-r}(M^t) \quad (= (0) \text{ if } p < r).$$

Since M^t is a torsion module, $I_0(M^t)$ contains a regular element of S ; thus $I_r(M)$ is the smallest nonzero Fitting ideal of M . Since M^t is a direct summand of M , we have also

$$\dim \text{proj}(M^t) \leq \dim \text{proj}(M) \leq 1$$

so there is an exact sequence

$$0 \rightarrow F_1 \rightarrow F_2 \rightarrow M^t \rightarrow 0$$

with free modules F_1, F_2 of finite ranks, say r_1, r_2 respectively. $r_1 \leq r_2$ since r_2 linear homogeneous equations in more than r_2 unknowns, with coefficients in S , always have a nontrivial solution in S . (Cf. [3, Proposition 6.1].) On the other hand, since $I_0(M^t) \neq (0)$, $r_1 \geq r_2$. Thus $r_1 = r_2$, and therefore $I_0(M^t)$ is a principal ideal, whose generator is a regular element (since $I_0(M^t)$ contains a regular element).

(i) \Rightarrow (ii). Let e_1, e_2, \dots, e_n generate M . Since S is local, there are relations

$$\sum_{j=1}^n a_{ij}e_j = 0 \quad (i = 1, 2, \dots, n - r)$$

with $a_{ij} \in S$, such that $I_r(M)$ is generated by the determinant D of the $(n-r) \times (n-r)$ matrix

$$A = (a_{ij}), \quad 1 \leq i, j \leq n - r.$$

Since some multiple of D is regular, so is D .

For $1 \leq h \leq n - r$, we can multiply the above relations by the cofactors of the h th column in A and add to obtain

$$De_h + \sum_{j=n-r+1}^n D_j e_j = 0$$

where the D_j are certain determinants which are in $I_r(M)$ and hence are divisible by D ; so we have the element

$$e_h + \sum_{j=n-r+1}^n (D_j/D)e_j$$

which is annihilated by D , and hence lies in M^t . It follows that the canonical images $\bar{e}_{n-r+1}, \bar{e}_{n-r+2}, \dots, \bar{e}_n$ of $e_{n-r+1}, e_{n-r+2}, \dots, e_n$ in M/M^t generate M/M^t over S .

If K is the total ring of fractions of S , then

$$M_{(K)} = M \otimes_S K \cong (M/M^t) \otimes_S K$$

since M^t is the kernel of the canonical map $M \rightarrow M_{(K)}$. Identifying M/M^t with an S -submodule of $M_{(K)}$ we see that the r elements $\bar{e}_{n-r+1}, \dots, \bar{e}_n$ generate $M_{(K)}$ over K . The Fitting ideals of $M_{(K)}$ over K are obtained from those of M by localization. So if

$$\sum_{j=n-r+1}^n c_j \bar{e}_j = 0$$

with c_j in $S \subseteq K$, then the ideal $(c_{n-r+1}, \dots, c_n)K$ is contained in $I_{r-1}(M_{(K)}) = (0)$, i.e. $c_{n-r+1} = \dots = c_n = 0$. Thus $\bar{e}_{n-r+1}, \dots, \bar{e}_n$ form a free basis of M/M^t .

For the remaining assertions, note first that the rows of the $(n-r) \times n$ matrix (a_{ij}) are linearly independent; for if

$$\sum_{i=1}^{n-r} x_i a_{ij} = 0 \quad (j = 1, 2, \dots, n)$$

with elements $x_i \in S$, then $Dx_i = 0$ for all i , i.e. $x_i = 0$. Now suppose that

$$\begin{array}{ccc}
 M & \xrightarrow{f} & M^{**} \\
 \downarrow & & \downarrow \mathfrak{R} \\
 M/M^t & \xrightarrow{g} & (M/M^t)^{**}
 \end{array}$$

If M/M^t is free then g is an isomorphism and so f is surjective. We see also from the above isomorphisms that then $M \otimes_S K$ is free over K and M^* is free of finite rank over S . Thus (i) \Rightarrow (ii).

Conversely if (ii) holds then M^{**} is free of finite rank; g is surjective since f is, and since $M \otimes_S K$ is free and $M/M^t \subseteq M \otimes_S K$, it is easily seen that g is also injective. Thus M/M^t is free. Q.E.D.

Applying Lemma 2 to the case $S=R$, $M=D$, we have that if D/D^t is free then D^* is free and $D \rightarrow D^{**}$ is surjective. But if k has characteristic zero, this implies that R is regular (cf. [3, §§3, 4]).

This completes the proof of the theorem.

REMARK. In Lemma 2, if S is noetherian and every prime ideal in S has depth ≤ 1 , then " M^* free of finite rank" implies that $f: M \rightarrow M^{**}$ is surjective (cf. [3, §6]). This gives the main Theorem 3 in [4].

REFERENCES

1. D. Ferrand, *Suite régulière et intersection complète*, C. R. Acad. Sci. Paris 264 (1967), 427-428.
2. H. Fitting, *Die Determinantenideale eines Moduls*, Jber. Deutsch. Math.-Verein. 46 (1936), 195-228.
3. J. Lipman, *Free derivation modules on algebraic varieties*, Amer. J. Math. 87 (1965), 874-898.
4. T. Matsuoka, *On the torsion submodule of a module of type (F_1)* , J. Sci. Hiroshima Univ. Ser. A-I Math. 31 (1967), 151-160.

PURDUE UNIVERSITY