IDEMPOTENT PROBABILITY MEASURES ON COMPACT SEMITOPOLOGICAL SEMIGROUPS

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The structure of idempotent probability measures on compact topological semigroups is well known (see, for example, [2], [4], [7] and [9]). However, the statement in [8] that the methods of [7] can be used to obtain identical results when the semigroup is only semitopological (i.e. the multiplication is separately rather than jointly continuous) is misleading, since the minimal ideal of a compact semitopological semigroup may not be closed [1, Chapter IV, Example 7.1]. In this note, it will be indicated how the convolution decomposition of a probability idempotent may be achieved in the semitopological case.

Let S be a compact semitopological semigroup. For each continuous function f on S and each pair x, y of elements of S, we write ${}_xf(y)=f(xy)$ and $f_x(y)=f(yx)$. If ν is a Radon measure on S we put $\nu(f)=\int f(x)d\nu(x)$, and define ${}_\nu f(x)=\nu(f_x)$ and $f_\nu(x)=\nu({}_xf)$; the functions ${}_\nu f$ and f_ν are continuous (see [3]). If λ is any other Radon measure on S, the mapping $(x, y)\to f(xy)$ is measurable with respect to the product measure $\lambda\otimes\nu$ (see [5] or [6]), and (using the Fubini Theorem) we may define the convolution product $\lambda*\nu$ by writing

$$\lambda * \nu(f) = \int f(xy)d\lambda \otimes \nu(x, y) = \lambda(f_\nu) = \nu({}_\lambda f).$$

It is well known (see [3]) that the space of measures becomes an algebra under convolution.

We take a fixed idempotent probability measure μ; thus, $\mu*\mu=\mu$, $\mu\geqq 0$, and $\|\mu\|=\mu(1)=1$. As the methods we use are familiar, we shall only sketch the proofs. The first step is to describe the support of a convolution product, and the result is Lemma 4.1 of [3].

LEMMA 1. *If* λ *and* ν *are positive measures, the support* $\operatorname{supp}(\lambda*\nu)$ *of* $\lambda*\nu$ *is the closure of the product* $\operatorname{supp}\lambda\cdot\operatorname{supp}\nu$ *of the supports of* λ *and* ν.

It follows that $\operatorname{supp}\mu$ is a semigroup. Therefore we need only consider the restriction of μ to $\operatorname{supp}\mu$, and to simplify our notation we shall assume $S=\operatorname{supp}\mu$. For each continuous function f on S, we write ${}_\mu f_\mu={}_\mu(f_\mu)=({}_\mu f)_\mu$.

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LEMMA 2. (i) *If* $f \geqq 0$ *and* ${}_\mu f_\mu = 0$, *then* $f = 0$.
(ii) ${}_\mu f_\mu$ *is constant.*
(iii) *The minimal ideal K of S is dense in S.*

The proofs are modeled on some of Glicksberg's [3], [7].

(i) From Lemma 1, $S.S$, and so $S.S.S$, is dense in S. If $f \geqq 0$ and $f \neq 0$, we can find x, y, z in S such that $f(xyz) > 0$, whence ${}_\mu f_\mu(y) = \int\int f(xyz)d\mu(x)d\mu(z) > 0$.

(ii) Let ${}_\mu f_\mu$ attain its supremum at $a \in S$. Then

$$ {}_\mu f_\mu(a) = {}_{\mu * \mu} f_{\mu * \mu}(a) = \int\int {}_\mu f_\mu(xay)d\mu(x)d\mu(y) \leqq \int\int {}_\mu f_\mu(a)d\mu(x)d\mu(y) $$
$$ = {}_\mu f_\mu(a), $$

whence ${}_\mu f_\mu(a) = {}_\mu f_\mu(xay)$ $(x, y \in S)$. So ${}_\mu f_\mu$ attains its supremum everywhere on $SaS \supseteq K$. By symmetry, ${}_\mu f_\mu$ attains its infimum on K, and as $K \neq \emptyset$, ${}_\mu f_\mu$ is constant.

(iii) If $f \geqq 0$ and f vanishes on K, then for $y \in K$, ${}_\mu f_\mu(y) = \int\int f(xyz)d\mu(x)d\mu(z) = 0$ since $xyz \in K$. By (i) and (ii) $f = 0$. Thus K is dense in S.

LEMMA 3. *For any probability measure* ν, $\mu * \nu * \mu = \mu$.

Let f be continuous, and let k be the constant value of ${}_\mu f_\mu$. Then $\mu * \nu * \mu(f) = \nu({}_\mu f_\mu) = \nu(k) = k$, which is independent of ν. Thus, $\mu * \nu * \mu = \mu * \mu * \mu = \mu$.

We must now describe the structure of the minimal ideal K of S. Proofs of our assertions may be found in [1, see in particular Theorem 3.5 of Chapter II].

There must exist an idempotent e in K. Then $Se = Ke$ is a minimal left ideal of S, while $eS = eK$ is a minimal right ideal; $eSe = eKe$ is a group, which we will denote by G. Each of these three subsemigroups of S is compact, and multiplication in each is jointly continuous; G is even a topological group. The set E of idempotents in Ke is compact, and is a subsemigroup of Ke with multiplication $e_1e_2 = e_1$. The corresponding subsemigroup F of idempotents in eK is compact with multiplication $f_1f_2 = f_2$. The product EGF in S is equal to K, and each x in K has a unique expression in the form $x = e'x'f'$ with e' in E, x' in G and f' in F. Because $Fe \subseteq eKe = G$, $Ke = EGFe = EG$. In the same way, we see that $eK = GF$.

THEOREM. *The measure μ can be expressed in the form $\mu_E * \mu_G * \mu_F$ where μ_E has support E, μ_G is the (normalized) Haar measure of G, and μ_F has support F.*

Let δ_e be the measure defined by $\delta_e(f)=f(e)$. First we use Lemma 3 (taking $\nu=\delta_e$) to obtain $\mu * \delta_e * \mu * \delta_e=\mu * \delta_e$, so that $\mu * \delta_e$ is idempotent. According to Lemma 1, the support of $\mu * \delta_e$ is the closure of Se; we have observed above that $Se=EG$ is itself compact and is a jointly continuous semigroup. We may apply known results (specifically, Proposition 3.1 §C of [7]) to conclude that $\mu * \delta_e=\mu_E * \mu_G$ where μ_E has support E and μ_G is the (normalized) Haar measure of G. A parallel argument proves that $\delta_e * \mu=\mu_G * \mu_F$, where the support of μ_F is F. Applying Lemma 3 again with $\nu=\delta_e * \delta_e=\delta_e$, we obtain

$$\mu = \mu * \delta_e * \delta_e * \mu = \mu_E * \mu_G * \mu_G * \mu_F = \mu_E * \mu_G * \mu_F,$$

for μ_G, being the Haar measure of G, is itself idempotent.

References

1. J. F. Berglund and K. H. Hofmann, *Compact semitopological semigroups and weakly almost periodic functions*, Lecture Notes in Mathematics 42, Springer-Verlag, Berlin, 1967.

2. H. S. Collins, *Idempotent measures on compact semigroups*, Proc. Amer. Math. Soc. **13** (1962), 442–446.

3. I. Glicksberg, *Weak compactness and separate continuity*, Pacific J. Math. **11** (1961), 205–214.

4. M. Heble and M. Rosenblatt, *Idempotent measures on a compact topological semigroup*, Proc. Amer. Math. Soc. **14** (1963), 177–184.

5. B. E. Johnson, *Separate continuity and measurability*, Proc. Amer. Math. Soc. **20** (1969), 420–422.

6. W. Moran, *Separate continuity and supports of measures*, J. London Math. Soc. **44** (1969), 320–324.

7. J. S. Pym, *Idempotent measures on semigroups*, Pacific J. Math. **12** (1962), 685–698.

8. ———, *Weakly separately continuous measure algebras*, Math. Ann. **175** (1968), 207–219.

9. S. Schwarz, *Product decomposition of idempotent measures on compact semigroups*, Czechoslovak Math. J. **14** (1964), 121–124.

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