ADDENDA AND CORRIGENDA TO "ON FILIPPOV'S IMPLICIT FUNCTIONS LEMMA"

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1. The authors have had their attention called to a previously published note by C. Castaing [2] containing results that overlap considerably with Theorem 1.

2. A paper [3] by L. Cesari that appeared after proof-reading of [1] motivated the following strengthening and simplification of Theorem 2 of [1]; the connection between references [3] and [4] and the theorem will be explained after the proof.

THEOREM 2'. Let $C^*$ be the union of countably many compact metrizable sets. For each $(x, t)$ in $\mathbb{R}^{n+1}$ let $C(x, t)$ be a subset of $C^*$ such that the set $M^*$ of all $(x, t, v)$ with $(x, t)$ in $\mathbb{R}^{n+1}$ and $v$ in $C(x, t)$ is a closed subset of $\mathbb{R}^{n+1} \times C^*$. Let $f^1, \ldots, f^n$ be continuous real-valued functions on $M^*$. Let $x: [a, b] \rightarrow \mathbb{R}$ be an absolutely continuous function such that for almost all $t$ in $[a, b]$, $x'(t)$ is contained in the convex cover of the image $f(x(t), t, C(x(t), t))$ of $C(x(t), t)$ in $\mathbb{R}^n$. Then there exist $n+1$ measurable functions $v_i: [a, b] \rightarrow C^*$ and $n+1$ measurable nonnegative functions $p_i: [a, b] \rightarrow \mathbb{R}$ such that for all $t$ in $[a, b]$, each $v_i(t)$ is in $C(x(t), t)$, and

$$\sum_{i=1}^{n+1} p_i(t) = 1$$

and for almost all $t$ in $[a, b]$

$$x''(t) = \sum_{i=1}^{n+1} p_i(t)f^i(x(t), t, v_i(t)).$$

Let $W_{n+1}$ be the set of all $(n+1)$-tuples $(p_1, \ldots, p_{n+1})$ with all $p_j \geq 0$ and $\sum p_j = 1$. Then the set

$$Q = (M^*)^{n+1} \times W_{n+1}$$

is the union of countably many metrizable compact sets. Let $k$ be the mapping from $Q$ into $\mathbb{R}^{n+1}$ whose value at the point

$$s = (x_1, l_1, v_1, \ldots, x_{n+1}, l_{n+1}, v_{n+1}, p_1, \ldots, p_{n+1})$$

is given by

$$k^i(s) = \sum_{j=1}^{n+1} p_j f^i(x_j, t_j, v_j) \quad (i = 1, \ldots, n),$$

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\begin{align*}
k^{n+i}(z) &= x^i_j \quad (j = 1, \ldots, n+1; \ i = 1, \ldots, n), \\
k^{n+2m+i} &= l_j \quad (j = 1, \ldots, n+1).
\end{align*}

This is continuous on \( Q \).

There is a subset \( M \) of \([a, b]\) with measure \( b-a \) such that for all \( t \) in \( M \), \( x'(t) \) exists and is in the smallest convex set that contains \( f(x(t), t, C(x(t), t)) \). By a theorem of Carathéodory it therefore can be written as

\[
x'(t) = \sum_{j=1}^{n+1} p_j f^i(x(t), t, v_i)
\]

where \( p \) is in \( W_{n+1} \) and each \( v_j \) is in \( C(x(t), t) \). Therefore if in the expression (2) for \( s \) we choose each \( t_j \) to be \( t \) and each \( x_j \) to be \( x(t) \), we obtain

\[
k^i(z) = x'(t) \quad (i = 1, \ldots, n),
\]

\[
k^{n+i}(z) = x'(i) \quad (j = 1, \ldots, n+1; \ i = 1, \ldots, n),
\]

\[
k^{n+2m+i}(z) = t \quad (j = 1, \ldots, n+1).
\]

We define \( y: M \to R^{n+3n+1} \) by setting \( y(t) = (x'(t), x(t), \ldots, x(t), t, \ldots, t) \), the dots denoting \((n+1)\)-fold repetition. The preceding equations imply \( y(M) \subseteq k(Q) \). Hence, by Theorem 1, there exists a measurable function \( u: M \to Q \) such that

\[
y(u(t)) = y(t) \quad (t \text{ in } M).
\]

We denote \( u(t) \) by

\[
(x_1(t), t_1(t), v_1(t), \ldots, x_{n+1}(t), t_{n+1}(t), v_{n+1}(t), p_1(t), \ldots, p_{n+1}(t)).
\]

Then (3) implies that for \( i = 1, \ldots, n \) and \( j = 1, \ldots, n+1 \) we have

\[
\sum_{j=1}^{n+1} p_j f^i(x_j(t), t_j(t), v_j(t)) = x'(t),
\]

\[
x_j(t) = x^i(t),
\]

\[
t_j(t) = t.
\]

Substituting the last pair of equations in the one preceding then yields (1) for all \( t \) in \( M \), completing the proof.

The "chattering controls" of Gamkrelidze \([4]\) are the functions \((p_1, \ldots, p_{n+1}, v_1, \ldots, v_{n+1})\) of (1); for generalized curves based on such controls, Gamkrelidze established the maximum principle. Amending the definition by allowing \( v \) (\( \geq n+1 \)) components in \( p \) and
Cesari established the existence of an optimizing generalized curve. The generalized curves of Young and McShane replace (1) by

\[ x^* = \int f'(x(t), t, \nu)p_t(d\nu) \]

with \( p_t \) a probability measure on \( C(x(t), t) \). By Theorem 2', if the convex hull of \( f(x(t), t, C(x(t), t)) \) is closed for all \( t \), there is a chattering control in the sense of Gamkrelidze that yields the same trajectory; the different formulations are in effect interchangeable.

3. At the bottom of page 41, change \([0, \infty)\) to \((0, \infty)\).

4. The first line of Theorem 2 should read "If \( C^* \) is the union of a countable set \( K_1 \subseteq K_2 \subseteq K_3 \subseteq \cdots \)." However, the theorem is in fact correct even as misprinted, since this is a special case of Theorem 2'.

REFERENCES


University of Virginia and
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