ADDENDA AND CORRIGENDA TO "ON FILIPPOV'S IMPLICIT FUNCTIONS LEMMA"

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1. The authors have had their attention called to a previously published note by C. Castaing [2] containing results that overlap considerably with Theorem 1.

2. A paper [3] by L. Cesari that appeared after proof-reading of [1] motivated the following strengthening and simplification of Theorem 2 of [1]; the connection between references [3] and [4] and the theorem will be explained after the proof.

THEOREM 2′. *Let C^* be the union of countably many compact metrizable sets. For each (x, t) in R^{n+1} let $C(x, t)$ be a subset of C^* such that the set M^* of all (x, t, v) with (x, t) in R^{n+1} and v in $C(x, t)$ is a closed subset of $R^{n+1}\times C^*$. Let $f^1, \cdots, f^n$ be continuous real-valued functions on M^*. Let x: $[a, b] \to R$ be an absolutely continuous function such that for almost all t in $[a, b]$, $x'(t)$ is contained in the convex cover of the image $f(x(t), t, C(x(t), t))$ of $C(x(t), t)$ in R^n. Then there exist $n+1$ measurable functions v_j: $[a, b]\to C^*$ and $n+1$ measurable nonnegative functions p_j: $[a, b]\to R$ such that for all t in $[a, b]$, each $v_j(t)$ is in $C(x(t), t)$, and $\sum_{j=1}^{n+1} p_j(t) = 1$, and for almost all t in $[a, b]$*

$$x^{i\prime}(t) = \sum_{j=1}^{n+1} p_j(t) f^i(x(t), t, v_j(t)). \tag{1}$$

Let W_{n+1} be the set of all $(n+1)$-tuples $(p_1, \cdots, p_{n+1})$ with all $p_j \geqq 0$ and $\sum p_j = 1$. Then the set

$$Q = (M^*)^{n+1} \times W_{n+1}$$

is the union of countably many metrizable compact sets. Let k be the mapping from Q into R^{n^2+3n+1} whose value at the point

$$z = (x_1, t_1, v_1, \cdots, x_{n+1}, t_{n+1}, v_{n+1}, p_1, \cdots, p_{n+1}) \tag{2}$$

is given by

$$k^i(z) = \sum_{j=1}^{n+1} p_j f^i(x_j, t_j, v_j) \qquad (i = 1, \cdots, n),$$

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$$k^{jn+i}(z) = x_j^i \qquad (j = 1, \cdots, n+1; \quad i = 1, \cdots, n),$$
$$k^{n^2+2n+j} = t_j \qquad (j = 1, \cdots, n+1).$$

This is continuous on Q.

There is a subset M of $[a, b]$ with measure $b-a$ such that for all t in M, $x'(t)$ exists and is in the smallest convex set that contains $f(x(t), t, C(x(t), t))$. By a theorem of Carathéodory it therefore can be written as

$$x^{i'}(t) = \sum_{j=1}^{n+1} p_j f^i(x(t), t, v_j)$$

where p is in W_{n+1} and each v_j is in $C(x(t), t)$. Therefore if in the expression (2) for z we choose each t_j to be t and each x_j to be $x(t)$, we obtain

$$k^i(z) = x^{i'}(t) \qquad (i = 1, \cdots, n),$$
$$k^{jn+i}(z) = x^i(t) \qquad (j = 1, \cdots, n+1; \quad i = 1, \cdots, n),$$
$$k^{n^2+2n+j}(z) = t \qquad (j = 1, \cdots, n+1).$$

We define $y: M \to R^{n^2+3n+1}$ by setting $y(t) = (x'(t), x(t), \cdots, x(t), t, \cdots, t)$, the dots denoting $(n+1)$-fold repetition. The preceding equations imply $y(M) \subseteqq k(Q)$. Hence, by Theorem 1, there exists a measurable function $u: M \to Q$ such that

$$k(u(t)) = y(t) \qquad (t \text{ in } M). \tag{3}$$

We denote $u(t)$ by

$$(x_1(t), t_1(t), v_1(t), \cdots, x_{n+1}(t), t_{n+1}(t), v_{n+1}(t), p_1(t), \cdots, p_{n+1}(t)).$$

Then (3) implies that for $i = 1, \cdots, n$ and $j = 1, \cdots, n+1$ we have

$$\sum_{j=1}^{n+1} p_j(t) f^i(x_j(t), t_j(t), v_j(t)) = x^{i'}(t),$$
$$x_j^i(t) = x^i(t),$$
$$t_j(t) = t.$$

Substituting the last pair of equations in the one preceding then yields (1) for all t in M, completing the proof.

The "chattering controls" of Gamkrelidze [4] are the functions $(p_1, \cdots, p_{n+1}, v_1, \cdots, v_{n+1})$ of (1); for generalized curves based on such controls, Gamkrelidze established the maximum principle. Amending the definition by allowing ν $(\geqq n+1)$ components in p and

v, Cesari established the existence of an optimizing generalized curve. The generalized curves of Young and McShane replace (1) by

$$x^{i\prime} = \int f^i(x(t), t, v) p_t(dv)$$

with p_t a probability measure on $C(x(t), t)$. By Theorem 2′, if the convex hull of $f(x(t), t, C(x(t), t))$ is closed for all t, there is a chattering control in the sense of Gamkrelidze that yields the same trajectory; the different formulations are in effect interchangeable.

3. At the bottom of page 41, change $[0, \infty)$ to $(0, \infty)$.

4. The first line of Theorem 2 should read "If C^* is the union of a countable set $K_1 \subseteqq K_2 \subseteqq K_3 \subseteqq \cdots$." However, the theorem is in fact correct even as misprinted, since this is a special case of Theorem 2′.

References

1. E. J. McShane and R. B. Warfield, Jr., *On Filippov's implicit functions lemma*, Proc. Amer. Math. Soc. **18** (1967), 41–47.

2. C. Castaing, *Quelques problèmes de mesurabilité liés à la théorie de la commande*, C. R. Acad. Sci. Paris A**262** (1966), 409–411.

3. L. Cesari, *Existence theorems for weak and usual optimal solutions in Lagrange problems with unilateral constraints*. II: *Existence theorems for weak solutions*, Trans. Amer. Math. Soc. **124** (1966), 413–430.

4. R. V. Gamkrelidze, *On sliding optimal states*, Dokl. Akad. Nauk SSSR **143** (1962), 1243–1245 = Soviet Math. Dokl. **3** (1962), 559–561.

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