THE DIFFERENCE CONSTRUCTION IN K-THEORY

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Let G be a compact Lie group, H a closed subgroup, α and β real or complex representations of G which become equivalent when restricted to H. Then α, β and an equivalence on H give rise to an element in $K^{-1}(G/H)$ or $KO^{-1}(G/H)$ (depending upon whether the representations are complex or real). The question has been raised (see [3, p. 122]) whether this element, in the real case, is independent of the particular equivalence chosen. The purpose of this note is to show that in fact this element does depend upon the choice of equivalence in the real case.

We recall the construction. Let α and β be representations of G thought of as homomorphisms from G to $O(n)$ and σ an element of $O(n)$ such that $\alpha(h)=\sigma\beta(h)\sigma^{-1}$ for all h in H. Then we have a map from G/H to $O(n)$ given by

$$gH \to \alpha(g)\sigma\beta(g^{-1})\sigma^{-1}.$$

This map gives rise to an element of $KO^{-1}(G/H)$.

First we take $G=\mathrm{Spin}(8n+1)$, $H=\mathrm{Spin}(8n)$, $G/H=S^{8n}$ and α and β to both be the faithful irreducible $\mathrm{Spin}(8n+1)$ representation of dimension $N=2^{4n}$ (see [2] or [3, p. 64]), and we take σ to be the identity. Then the element we get in $KO^{-1}(S^{8n})$ is clearly zero.

Now, using the data of [2], $C_k=C_k^0+C_k^1$ is the Clifford algebra, $R^k\subset C_k$, $e_1, \cdots, e_k$ the standard basis for R^k. Let $M=M^0+M^1$ be the graded, irreducible C_{8n+1} representation whose first summand M^0 when restricted to $\mathrm{Spin}(8n+1)$ is the irreducible 2^{4n}-dimensional $\mathrm{Spin}(8n+1)$-representation α. The element $\epsilon=e_1\cdots e_{8n}$ in C_{8n+1}^0 has the property that $e_ie_j\epsilon=\epsilon e_ie_j$ for $1\leqq i\leqq j\leqq 8n$. The image of C_{8n}^0 in C_{8n+1}^0 is generated by monomials $e_{i_1}\cdots e_{i_{2k}}$ where $i_1<i_2<\cdots<i_{2k}\leqq 8n$, so ϵ commutes with elements in C_{8n}^0 and so commutes with elements of $\mathrm{Spin}(8n)$. Thus multiplication by ϵ is an automorphism of M^0 which commutes with the action of $\mathrm{Spin}(8n)$. Thus our construction gives rise to a map

$$f\colon \mathrm{Spin}(8n+1)/\mathrm{Spin}(8n) = S^{8n} \to O(M^0).$$

The coset of the element g in $\mathrm{Spin}(8n+1)$ is identified with the element $ge_{8n+1}g^{-1}$, and

$$f(ge_{8n+1}g^{-1}) = \alpha(g)\epsilon\alpha(g^{-1})\epsilon^{-1}$$

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or, thinking of $\mathrm{Spin}(8n+1)$ as acting on M^0,

$$f(ge_{8n+1}g^{-1}) = g\epsilon g^{-1}\epsilon^{-1}.$$

Since $\mathrm{Spin}(8n+1)$ is connected, we can connect ϵ^{-1} to the identity by a path ϵ_t. The map

$$f_t(ge_{8n+1}g^{-1}) = g\epsilon g^{-1}\epsilon_t$$

gives us a homotopy between f and

$$f'(ge_{8n+1}g^{-1}) = g\epsilon g^{-1}$$

and so both f and f' give rise to the same element of $KO^{-1}(S^{8n})$.

To identify this element, we look at the description of the generator of $KO^{-1}(S^{8n})$ given in [2]. We will denote the upper hemisphere of S^{8n+1} by S^+ and the lower hemisphere by S^-. A bundle E which is a generator of $KO^{-1}(S^{8n})=\tilde{K}O^0(S^{8n+1})$ is obtained by taking $S^+\times M^0$ and $S^-\times M^1$ and making the identification (x, m) in $S^{8n}\times M^0$ with (x, xm) in $S^{8n}\times M^1$. Here we think of x in $S^{8n}\subset C^1_{8n+1}$ so that $xM^0=M^1$. Now we will pick out the characteristic map of the bundle E. We let E^+ and E^- be the restrictions of E to S^+ and S^-. E is an $N=2^{4n}$-dimensional bundle and we will take our standard R^N to be M^0. Then we have maps

$$S^{8n}\times R^N = S^{8n}\times M^0 \xrightarrow{f_+} E^+ = S^{8n}\times M^0$$

and

$$S^{8n}\times R^N = S^{8n}\times M^0 \xrightarrow{f_-} E^- = S^{8n}\times M^1$$

given by $f_+=$identity and $f_-=$multiplication by $-e_1\cdots e_{8n+1}$ on the M^0-factor. As before, we take a generic element in S^{8n} to be $ge_{8n+1}g^{-1}$. Then

$$\begin{aligned}
f_-^{-1}f_+(ge_{8n+1}g^{-1}, m) &= f_-^{-1}(ge_{8n+1}g^{-1}, m)\\
&= f_-^{-1}(ge_{8n+1}g^{-1}, ge_{8n+1}g^{-1}m)\\
&= (ge_{8n+1}g^{-1}, -e_1\cdots e_{8n+1}ge_{8n+1}g^{-1}m).
\end{aligned}$$

Fortunately, $e_1\cdots e_{8n+1}$ is in the center of C_{8n+1} so

$$-e_1\cdots e_{8n+1}ge_{8n+1}g^{-1} = ge_1\cdots e_{8n}g^{-1} = g^{-1}\epsilon g.$$

Thus, the characteristic map of E is f' and so f' represents a generator of $KO^{-1}(S^{8n})$.

References

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