ON THE EXISTENCE OF $\mathfrak{c}$-POINTS IN $\beta N\backslash N$

NEIL HINDMAN[1]

R. S. Pierce has asked in [2] whether it is possible to show, without using the continuum hypothesis, that there are points of $\beta N\backslash N$ which lie simultaneously in the closure of three pairwise disjoint open sets. Such points are called, in his terminology, 3-points of $\beta N\backslash N$. In general, for any cardinal number $\mathfrak{n}$, and for any topological space X, a point x in X is called an $\mathfrak{n}$-point if it lies in the closure of each of $\mathfrak{n}$ pairwise disjoint open subsets of X.

It is shown here in §1, as a corollary to a more general theorem, that, without appeal to the continuum hypothesis, not only are there 3-points in $\beta N\backslash N$, but in fact there are $\mathfrak{c}$-points in $\beta N\backslash N$. It is further shown in §2 that, if the continuum hypothesis is assumed, *each* point of $\beta N\backslash N$ is a $\mathfrak{c}$-point. These results seem particularly striking since $\beta N\backslash N$ is an F'-space, i.e., disjoint cozero sets in $\beta N\backslash N$ have disjoint closures. (A cozero set in a space X is a preimage by a continuous real valued function of an open set.) It is well known that if α is any cardinal greater than $\mathfrak{c}$ then there do not exist α-points in $\beta N\backslash N$, since in fact, if $\mathfrak{U}$ is a collection of pairwise disjoint open sets in $\beta N\backslash N$ then $|\mathfrak{U}| \leqq \mathfrak{c}$.

1. Results without the continuum hypothesis. Let A^p be the ultrafilter on a discrete space D associated with the point p in βD. The ultrafilter A^p on D is called uniform if each element of A^p has cardinality $|D|$. Define $\mu D = \{p \in \beta D : A^p \text{ is uniform}\}$. Note that $\mu N = \beta N\backslash N$.

We include here for completeness the following facts about βD for discrete D. Our notation and general point of view are those of the Gillman and Jerison textbook [1], to which the reader is referred for proofs and additional discussion. For each $Z \subset D$, we have $\mathrm{cl}_{\beta D} Z = \{p \in \beta D : Z \in A^p\}$. Basic neighborhoods of a point p in $\beta D\backslash D$ are of the form $(\mathrm{cl}_{\beta D} Z)\backslash D$ where $Z \in A^p$.

1.1 LEMMA. *Let D be a discrete space of cardinality $\mathfrak{m}$, where $\mathfrak{m} \geqq \aleph_0$. Let $\mathfrak{A}$ be an infinite collection of subsets of D with $|A| = \mathfrak{m}$ for each*

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A in $\mathfrak{A}$ and with $|A_1 \cap A_2| < \mathfrak{m}$ for each distinct pair of elements A_1, A_2 of $\mathfrak{A}$. Then there is a uniform ultrafilter A^p on D such that $|\{A \in \mathfrak{A}: |Z \cap A| = \mathfrak{m}\}| = |\mathfrak{A}|$ for each Z in A^p.

PROOF. For each A in $\mathfrak{A}$ choose x_A in $\mathrm{cl}_{\beta D} A \cap \mu D$ and let $B = \{x_A: A \in \mathfrak{A}\}$. Now if $A_1 \neq A_2$ then $x_{A_1} \neq x_{A_2}$ since x_{A_1} and x_{A_2} are elements of μD and $|A_1 \cap A_2| < \mathfrak{m}$. Thus $|B| = |\mathfrak{A}|$. Now μD is closed in βD, hence is compact. (For if $p \in \beta D \setminus \mu D$ then there exists Z in A^p with $|Z| < \mathfrak{m}$, and $\mathrm{cl}_{\beta D} Z$ is a neighborhood of p missing μD.) Clearly then there is a point p in μD such that each neighborhood of p contains $|\mathfrak{A}|$ elements of B. (For if not, pick a neighborhood of each point of μD which contains less than $|\mathfrak{A}|$ elements of B. From the existence of a finite subcover it follows immediately that $|B| < |\mathfrak{A}|$, a contradiction.) But then A^p is the desired ultrafilter, for if $|Z \cap A| < \mathfrak{m}$ then $x_A \notin \mathrm{cl}_{\beta D} Z \cap \mu D$.

It follows from Theorem 7 of [3] that if $\mathfrak{m}$ is a cardinal number such that $2^{\mathfrak{n}} \leqq \mathfrak{m}$ for all $\mathfrak{n} < \mathfrak{m}$ and if D is a set of cardinality $\mathfrak{m}$, then there is a family $\mathfrak{A}$ of subsets of D, each of cardinality $\mathfrak{m}$, such that $|\mathfrak{A}| = 2^{\mathfrak{m}}$ and $|A_1 \cap A_2| < \mathfrak{m}$ for each pair A_1, A_2 of elements of $\mathfrak{A}$. Note that if the generalized continuum hypothesis holds then each infinite cardinal $\mathfrak{m}$ has the above mentioned property. At any rate there are cofinally many cardinals with this property. (For let $\mathfrak{n}_0$ be any cardinal and define inductively $\mathfrak{n}_{i+1} = 2^{\mathfrak{n}_i}$. Then $\mathfrak{m} = \sup\{\mathfrak{n}_i: i \in \omega_0\}$ has the given property and is greater than $\mathfrak{n}_0$.)

1.2 LEMMA. *Let D be a discrete space of cardinality $\mathfrak{m}$ where $2^{\mathfrak{n}} \leqq \mathfrak{m}$ for all $\mathfrak{n} < \mathfrak{m}$. Let $\mathfrak{A}$ be a collection of subsets of D as given above. Let A^p be a uniform ultrafilter as given by Lemma 1.1, the elements Z_α of A^p being indexed by the ordinals less than $2^{\mathfrak{m}}$. Then it is possible to choose, for each $\alpha < 2^{\mathfrak{m}}$, a subset X_α of Z_α so that $|X_\alpha| = \mathfrak{m}$ and so that $|X_\alpha \cap X_\gamma| < \mathfrak{m}$ whenever $\alpha < \gamma < 2^{\mathfrak{m}}$.*

PROOF. For Z_1 pick A_1 in $\mathfrak{A}$ such that $|Z_1 \cap A_1| = \mathfrak{m}$. Assume that for $\sigma < \alpha$ we have chosen $A_\sigma \in \mathfrak{A}$ such that $|Z_\sigma \cap A_\sigma| = \mathfrak{m}$ and $A_\sigma \neq A_\gamma$ for all $\gamma < \sigma$. Now $|\{A \in \mathfrak{A}: |A \cap Z_\alpha| \mathfrak{m}\}| = 2^{\mathfrak{m}}$ and $\alpha < 2^{\mathfrak{m}}$ so there is $A_\alpha \in \mathfrak{A}$ such that $|A_\alpha \cap Z_\alpha| = \mathfrak{m}$ and $A_\sigma \neq A_\alpha$ for all $\sigma < \alpha$. Let $X_\alpha = A_\alpha \cap Z_\alpha$.

1.3 THEOREM. *Let D be a discrete space of cardinality $\mathfrak{m}$ where $2^{\mathfrak{n}} \leqq \mathfrak{m}$ for all $\mathfrak{n} < \mathfrak{m}$. Then there is a $2^{\mathfrak{m}}$-point (i.e. a point in the closure of each of $2^{\mathfrak{m}}$ pairwise disjoint open sets) in μD.*

PROOF. Let $\mathfrak{A}$, $A^p = \{Z_\alpha\}_{\alpha < 2^{\mathfrak{m}}}$ and $X_\alpha \subset Z_\alpha$ be as in Lemma 1.2. Now $|X_\alpha| = \mathfrak{m}$ so there is a collection $\{X_{\tau\alpha}\}_{\tau < 2^{\mathfrak{m}}}$ of subsets of X_α

such that $|X_{\tau\alpha}| = \mathfrak{m}$ for all $\tau < 2^{\mathfrak{m}}$ and such that $|X_{\tau\alpha} \cap X_{\sigma\alpha}| < \mathfrak{m}$ if $\tau \neq \sigma$. For each $\tau < 2^{\mathfrak{m}}$ let $U_\tau = (\bigcup_{\alpha<2^{\mathfrak{m}}} \mathrm{cl}_{\beta D} X_{\tau\alpha}) \cap \mu D$. Recall that $\mathrm{cl}_{\beta D} X_{\tau\alpha}$ is open in βD, so that U_τ is open. We assert that the family $\{U_\tau\}_{\tau<2^{\mathfrak{m}}}$ is as desired. That is, it is a pairwise disjoint family and the closure of each member contains p. Suppose first that $\sigma \neq \tau$ and $q \in U_\sigma \cap U_\tau$. Then there is a pair α_1, α_2 such that $q \in \mathrm{cl}_{\beta D} X_{\sigma\alpha_1} \cap \mathrm{cl}_{\beta D} X_{\tau\alpha_2} \cap \mu D$. If $\alpha_1 \neq \alpha_2$ then $X_{\sigma\alpha_1} \cap X_{\tau\alpha_2} \subset X_{\alpha_1} \cap X_{\alpha_2}$ and $|X_{\alpha_1} \cap X_{\alpha_2}| < \mathfrak{m}$. If $\alpha_1 = \alpha_2$ then $|X_{\sigma\alpha_1} \cap X_{\tau\alpha_2}| < \mathfrak{m}$. So in fact, in either case $|X_{\sigma\alpha_1} \cap X_{\tau\alpha_2}| < \mathfrak{m}$. But since $q \in \mathrm{cl}_{\beta D} X_{\sigma\alpha_1} \cap \mathrm{cl}_{\beta D} X_{\tau\alpha_2}$ we have that $X_{\sigma\alpha_1} \cap X_{\tau\alpha_2} \in A^q$, contrary to the fact $q \in \mu D$.

Finally let $Z_\alpha \in A^p$. Now $\mathrm{cl}_{\beta D} Z_\alpha \cap \mu D$ is a basic neighborhood of p in μD so we need only show that $\mathrm{cl}_{\beta D} Z_\alpha \cap \mu D \cap U_\tau \neq \emptyset$ for all $\tau < 2^{\mathfrak{m}}$. Since $|X_{\tau\alpha}| = \mathfrak{m}$ and $X_{\tau\alpha} \subset X_\alpha \subset Z_\alpha$ it follows that $|X_{\tau\alpha} \cap Z_\alpha| = \mathfrak{m}$, so there is $q \in \mathrm{cl}_{\beta D} Z_\alpha \cap \mathrm{cl}_{\beta D} X_{\tau\alpha} \cap \mu D \subset \mathrm{cl}_{\beta D} Z_\alpha \cap U_\tau \cap \mu D \neq \emptyset$.

1.4 COROLLARY. *There is a* $\mathfrak{c}$*-point in* $\beta N\backslash N$.

PROOF. $2^{\mathfrak{n}} \leqq \aleph_0$ for all $\mathfrak{n} < \aleph_0$ and $\beta N\backslash N = \mu N$.

2. **Results assuming the continuum hypothesis.** We show here that, if the continuum hypothesis is assumed, not only do there exist $\mathfrak{c}$-points of $\beta N\backslash N$ as shown above, but in fact *each* point of $\beta N\backslash N$ is a $\mathfrak{c}$-point of $\beta N\backslash N$.

2.1 LEMMA. *Assume the continuum hypothesis and let* A^p *be a free (i.e. uniform) ultrafilter on* N. *Order the elements of* A^p *by the ordinals less than* ω_1. *Then there is a choice of* $X_\alpha \subset Z_\alpha$ *for each* $Z_\alpha \in A^p$ *such that* $|X_\alpha| = \aleph_0$ *and* $|X_\alpha \cap X_\beta| < \aleph_0$ *if* $\alpha \neq \beta$.

PROOF. Let X_1 be any infinite subset of Z_1 which is not in A^p. Let $\alpha < \omega_1$ and assume that for each $\sigma < \alpha$ we have chosen $X_\sigma \subset Z_\sigma$ so that $X_\sigma \notin A^p$ and $|X_\sigma \cap X_\gamma| < \aleph_0$ for all $\gamma < \sigma$. Let $B = \{\sigma < \alpha : |X_\sigma \cap Z_\alpha| = \aleph_0$. If B is finite then $(Z_\alpha \backslash \bigcup_{\sigma \in B} X_\sigma) \in A^p$ and we may let X_α be any infinite subset of $Z_\alpha \backslash \bigcup_{\sigma \in B} X_\sigma$ which is not in A^p. Then for $\sigma < \alpha$ we have $|X_\sigma \cap X_\alpha| < \aleph_0$. If B is infinite it is countable so we may write $B = \{\sigma_n\}_{n=1}^{\infty}$. Let $x_{\alpha k} \in (X_{\sigma_k} \cap Z_\alpha) \backslash \bigcup_{j<k} X_{\sigma_j}$. This is possible since $|X_{\sigma_k} \cap Z_\alpha| = \aleph_0$ and $|X_{\sigma_j} \cap X_{\sigma_k}| < \aleph_0$ for $j < k$. Let X_α be any infinite subset of $\{x_{\alpha k}\}_{k=1}^{\infty}$ which is not an element of A^p. Then $|X_\alpha \cap X_{\sigma_k}| \leqq k < \aleph_0$, and if $\sigma \notin B$ then $|X_\sigma \cap X_\alpha| \leqq |X_\sigma \cap Z_\alpha| < \aleph_0$.

2.2 THEOREM. *Assume the continuum hypothesis and let* $p \in \beta N\backslash N$. *Then* p *is a* $\mathfrak{c}$*-point of* $\beta N\backslash N$.

PROOF. Let $X_\alpha \subset Z_\alpha$ be as in Lemma 2.1 for each Z_α in A^p. The rest of the proof is the same as that of Theorem 1.3.

3. **Remarks.** It is an easy consequence of Theorem 1 of [3] that for any discrete space D of cardinality $\mathfrak{m}$ where $\mathfrak{m} \geqq \aleph_0$, μD does not have $\mathfrak{n}$-points for any $\mathfrak{n} > 2^{\mathfrak{m}}$ since in fact μD does not have $\mathfrak{n}$ pairwise disjoint open sets.

It is not possible to directly generalize the proof of Lemma 2.1 to obtain the corresponding statement about μD for discrete spaces D of higher cardinality. Thus the following question remains open: Even assuming the generalized continuum hypothesis, is each point of D a $2^{|D|}$-point for an infinite discrete space D?

The author has also been unable to determine if the continuum hypothesis is in fact a necessary condition for Theorem 2.2.

References

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Wesleyan University