ON THE EXISTENCE OF \( c \)-POINTS IN \( \beta \mathbb{N} \setminus \mathbb{N} \)

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R. S. Pierce has asked in [2] whether it is possible to show, without using the continuum hypothesis, that there are points of \( \beta \mathbb{N} \setminus \mathbb{N} \) which lie simultaneously in the closure of three pairwise disjoint open sets. Such points are called, in his terminology, 3-points of \( \beta \mathbb{N} \setminus \mathbb{N} \). In general, for any cardinal number \( n \), and for any topological space \( X \), a point \( x \) in \( X \) is called an \( n \)-point if it lies in the closure of each of \( n \) pairwise disjoint open subsets of \( X \).

It is shown here in §1, as a corollary to a more general theorem, that, without appeal to the continuum hypothesis, not only are there 3-points in \( \beta \mathbb{N} \setminus \mathbb{N} \), but in fact there are \( c \)-points in \( \beta \mathbb{N} \setminus \mathbb{N} \). It is further shown in §2 that, if the continuum hypothesis is assumed, each point of \( \beta \mathbb{N} \setminus \mathbb{N} \) is a \( c \)-point. These results seem particularly striking since \( \beta \mathbb{N} \setminus \mathbb{N} \) is an \( F' \)-space, i.e., disjoint cozero sets in \( \beta \mathbb{N} \setminus \mathbb{N} \) have disjoint closures. (A cozero set in a space \( X \) is a preimage by a continuous real valued function of an open set.) It is well known that if \( \alpha \) is any cardinal greater than \( c \) then there do not exist \( \alpha \)-points in \( \beta \mathbb{N} \setminus \mathbb{N} \), since in fact, if \( \mathcal{U} \) is a collection of pairwise disjoint open sets in \( \beta \mathbb{N} \setminus \mathbb{N} \) then \( \left| \mathcal{U} \right| \leq c \).

1. Results without the continuum hypothesis. Let \( A^\ast \) be the ultrafilter on a discrete space \( D \) associated with the point \( p \) in \( \beta D \). The ultrafilter \( A^\ast \) on \( D \) is called uniform if each element of \( A^\ast \) has cardinality \( |D| \). Define \( \mu D = \{ p \in \beta D : A^\ast \) is uniform \}. Note that \( \mu N = \beta \mathbb{N} \setminus \mathbb{N} \).

We include here for completeness the following facts about \( \beta D \) for discrete \( D \). Our notation and general point of view are those of the Gillman and Jerison textbook [1], to which the reader is referred for proofs and additional discussion. For each \( Z \subseteq D \), we have \( cl_D Z = \{ p \in \beta D : Z \subseteq A^\ast \} \). Basic neighborhoods of a point \( p \) in \( \beta D \setminus D \) are of the form \( (cl_D Z) \setminus D \) where \( Z \subseteq A^\ast \).

1.1 Lemma. Let \( D \) be a discrete space of cardinality \( m \), where \( m \geq \aleph_0 \). Let \( \mathcal{A} \) be an infinite collection of subsets of \( D \) with \( |A| = m \) for each.
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A in $\mathfrak{A}$ and with $|A_1 \cap A_2| < m$ for each distinct pair of elements $A_1, A_2$ of $\mathfrak{A}$. Then there is a uniform ultrafilter $A^p$ on $D$ such that $|\{A \in \mathfrak{A} : |Z \cap A| = m\}| = |\mathfrak{A}|$ for each $Z$ in $A^p$.

**Proof.** For each $A$ in $\mathfrak{A}$ choose $x_A$ in $\text{cl}_D A \cap \mu D$ and let $B = \{x_A : A \in \mathfrak{A}\}$. Now if $A_1 \neq A_2$ then $x_{A_1} \neq x_{A_2}$ since $x_{A_{\mathfrak{A}}}$ and $x_{A_{\mathfrak{A}}}$ are elements of $\mu D$ and $|A_1 \cap A_2| < m$. Thus $|B| = |\mathfrak{A}|$. Now $\mu D$ is closed in $\beta D$, hence is compact. (For if $p \in \beta D \setminus \mu D$ then there exists $Z$ in $A^p$ with $|Z| < m$, and $\text{cl}_D Z$ is a neighborhood of $p$ missing $\mu D$.) Clearly then there is a point $p$ in $\mu D$ such that each neighborhood of $p$ contains $|\mathfrak{A}|$ elements of $B$. (For if not, pick a neighborhood of each point of $\mu D$ which contains less than $|\mathfrak{A}|$ elements of $B$. From the existence of a finite subcover it follows immediately that $|Z| < |\mathfrak{A}|$, a contradiction.) But then $A^p$ is the desired ultrafilter, for if $|Z \cap A| < m$ then $x_A \in \text{cl}_D Z \cap \mu D$.

It follows from Theorem 7 of [3] that if $m$ is a cardinal number such that $2^n \leq m$ for all $n < m$ and if $D$ is a set of cardinality $m$, then there is a family $\mathfrak{A}$ of subsets of $D$, each of cardinality $m$ such that $|\mathfrak{A}| = 2^m$ and $|A_1 \cap A_2| < m$ for each pair $A_1, A_2$ of elements of $\mathfrak{A}$. Note that if the generalized continuum hypothesis holds then each infinite cardinal $m$ has the above mentioned property. At any rate there are cofinally many cardinals with this property. (For let $n_0$ be any cardinal and define inductively $n_{i+1} = 2^{n_i}$. Then $m = \sup \{n_i : i \in \omega_0\}$ has the given property and is greater than $n_0$.)

1.2 Lemma. Let $D$ be a discrete space of cardinality $m$ where $2^n \leq m$ for all $n < m$. Let $\mathfrak{A}$ be a collection of subsets of $D$ as given above. Let $A^p$ be a uniform ultrafilter as given by Lemma 1.1, the elements $Z_\alpha$ of $A^p$ being indexed by the ordinals less than $2^m$. Then it is possible to choose, for each $\alpha < 2^m$, a subset $X_\alpha$ of $Z_\alpha$ so that $|X_\alpha| = m$ and so that $|X_\alpha \cap X_\gamma| < m$ whenever $\alpha < \gamma < 2^m$.

**Proof.** For $Z_1$ pick $A_1$ in $\mathfrak{A}$ such that $|Z_1 \cap A_1| = m$. Assume that for $\sigma < \alpha$ we have chosen $A_\sigma \in \mathfrak{A}$ such that $|Z_\sigma \cap A_\sigma| = m$ and $A_\sigma \neq A_\gamma$ for all $\gamma < \sigma$. Now $|\{A \in \mathfrak{A} : |A \cap Z_\sigma| = m\}| = 2^m$ and $\alpha < 2^m$ so there is $A_\alpha \in \mathfrak{A}$ such that $|A_\alpha \cap Z_\alpha| = m$ and $A_\sigma \neq A_\alpha$ for all $\sigma < \alpha$. Let $X_\alpha = A_\alpha \cap Z_\alpha$.

1.3 Theorem. Let $D$ be a discrete space of cardinality $m$ where $2^n \leq m$ for all $n < m$. Then there is a $2^m$-point (i.e. a point in the closure of each of $2^m$ pairwise disjoint open sets) in $\mu D$.

**Proof.** Let $\mathfrak{A}$, $A^p = \{Z_\alpha\}_{\alpha < 2^m}$ and $X_\alpha \subset Z_\alpha$ be as in Lemma 1.2. Now $|X_\alpha| = m$ so there is a collection $\{X_{\alpha r}\}_{r < 2^m}$ of subsets of $X_\alpha$.
such that \(|X_{\tau}| = m\) for all \(\tau < 2^m\) and such that \(|X_{\tau} \cap X_{\sigma}| < m\) if \(\tau \neq \sigma\). For each \(\tau < 2^m\) let \(U_\tau = (\bigcup_{\tau < \sigma} \text{cl}_{\beta \mathbb{D}} X_{\sigma}) \cap \mu \mathbb{D}\). Recall that \(\text{cl}_{\beta \mathbb{D}} X_{\tau}\) is open in \(\beta \mathbb{D}\), so that \(U_\tau\) is open. We assert that the family \(\{U_\tau\}_{\tau < \sigma}\) is as desired. That is, it is a pairwise disjoint family and the closure of each member contains \(p\). Suppose first that \(\tau \neq \sigma\) and \(q \in U_\tau \cap U_\sigma\). Then there is a pair \(\alpha_1, \alpha_2\) such that \(q \in \text{cl}_{\beta \mathbb{D}} X_{\alpha_1} \cap \text{cl}_{\beta \mathbb{D}} X_{\alpha_2} \cap \mu \mathbb{D}\). If \(\alpha_1 \neq \alpha_2\) then \(X_{\alpha_1} \cap X_{\alpha_2} \subseteq X_{\alpha_1} \cap X_{\alpha_2}\) and \(|X_{\alpha_1} \cap X_{\alpha_2}| < m\). If \(\alpha_1 = \alpha_2\) then \(|X_{\alpha_1} \cap X_{\alpha_2}| < m\). So in fact, in either case \(|X_{\alpha_1} \cap X_{\alpha_2}| < m\). But since \(q \in \text{cl}_{\beta \mathbb{D}} X_{\alpha_1} \cap \text{cl}_{\beta \mathbb{D}} X_{\alpha_2}\) we have that \(X_{\alpha_1} \cap X_{\alpha_2} \in A_0\), contrary to the fact \(q \in \mu \mathbb{D}\).

Finally let \(Z_\alpha \in A_0\). Now \(\text{cl}_{\beta \mathbb{D}} Z_\alpha \cap \mu \mathbb{D}\) is a basic neighborhood of \(p\) in \(\mu \mathbb{D}\) so we need only show that \(\text{cl}_{\beta \mathbb{D}} Z_\alpha \cap \mu \mathbb{D} \cap U_\tau \neq \emptyset\) for all \(\tau < 2^m\).

Since \(|X_{\tau}| = m\) and \(X_{\tau} \subseteq X_{\alpha} \subseteq Z_\alpha\) it follows that \(|X_{\tau} \cap Z_\alpha| = m\), so there is \(q \in \text{cl}_{\beta \mathbb{D}} X_{\alpha} \cap \text{cl}_{\beta \mathbb{D}} X_{\tau} \cap \mu \mathbb{D} \cap \text{cl}_{\beta \mathbb{D}} Z_\alpha \cap U_\tau \cap \mu \mathbb{D} \neq \emptyset\).

**1.4 Corollary.** There is a c-point in \(\beta \mathbb{N} \setminus \mathbb{N}\).

**Proof.** \(2^n \leq \aleph_0\) for all \(n < \aleph_0\) and \(\beta \mathbb{N} \setminus \mathbb{N} = \mu \mathbb{N}\).

2. Results assuming the continuum hypothesis. We show here that, if the continuum hypothesis is assumed, not only do there exist c-points of \(\beta \mathbb{N} \setminus \mathbb{N}\) as shown above, but in fact each point of \(\beta \mathbb{N} \setminus \mathbb{N}\) is a c-point of \(\beta \mathbb{N} \setminus \mathbb{N}\).

**2.1 Lemma.** Assume the continuum hypothesis and let \(A^\mathbb{N}\) be a free (i.e. uniform) ultrafilter on \(\mathbb{N}\). Order the elements of \(A^\mathbb{N}\) by the ordinals less than \(\omega_1\). Then there is a choice of \(X_\alpha \in Z_\alpha\) for each \(Z_\alpha \in A^\mathbb{N}\) such that \(|X_\alpha| = \aleph_0\) and \(|X_\alpha \cap X_\beta| < \aleph_0\) if \(\alpha \neq \beta\).

**Proof.** Let \(X_1\) be any infinite subset of \(Z_1\) which is not in \(A^\mathbb{N}\). Let \(\alpha < \omega_1\) and assume that for each \(\sigma < \alpha\) we have chosen \(X_\sigma \subseteq Z_\sigma\) so that \(X_\sigma \in A^\mathbb{N}\) and \(|X_\sigma \cap X_\gamma| < \aleph_0\) for all \(\gamma < \sigma\). Let \(B = \{\sigma < \alpha\} : |X_\sigma \cap Z_\alpha| = \aleph_0\)\. If \(B\) is finite then \((Z_\alpha \setminus \bigcup_{\sigma \in B} X_\sigma) \in A^\mathbb{N}\) and we may let \(X_\alpha\) be any infinite subset of \((Z_\alpha \setminus \bigcup_{\sigma \in B} X_\sigma)\) which is not in \(A^\mathbb{N}\). Then for \(\sigma < \alpha\) we have \(|X_\sigma \cap X_\alpha| < \aleph_0\). If \(B\) is infinite it is countable so we may write \(B = \{\sigma_\alpha\}^{\omega_1}_{\alpha=1}\). Let \(x_\alpha \in (X_{\sigma_\alpha} \cap Z_\alpha) \setminus \bigcup_{j < k} X_{\sigma_j}\). This is possible since \(|X_{\sigma_\alpha} \cap Z_\alpha| = \aleph_0\) and \(|X_{\sigma_j} \cap X_{\sigma_k}| < \aleph_0\) for \(j < k\). Let \(X_\alpha\) be any infinite subset of \(|x_\alpha|^{\omega_1}_{\alpha=1}\) which is not an element of \(A^\mathbb{N}\). Then \(|X_\alpha \cap X_{\sigma_\alpha}| \leq k < \aleph_0\) and if \(\sigma \in B\) then \(|X_{\sigma} \setminus X_\alpha| \leq |X_{\sigma} \cap Z_\alpha| < \aleph_0\).

**2.2 Theorem.** Assume the continuum hypothesis and let \(p \in \beta \mathbb{N} \setminus \mathbb{N}\). Then \(p\) is a c-point of \(\beta \mathbb{N} \setminus \mathbb{N}\).

**Proof.** Let \(X_\alpha \subseteq Z_\alpha\) be as in Lemma 2.1 for each \(Z_\alpha\) in \(A^\mathbb{N}\). The rest of the proof is the same as that of Theorem 1.3.
3. Remarks. It is an easy consequence of Theorem 1 of [3] that for any discrete space $D$ of cardinality $m$ where $m \geq \aleph_0$, $\mu D$ does not have $n$-points for any $n > 2^m$ since in fact $\mu D$ does not have $n$ pairwise disjoint open sets.

It is not possible to directly generalize the proof of Lemma 2.1 to obtain the corresponding statement about $\mu D$ for discrete spaces $D$ of higher cardinality. Thus the following question remains open: Even assuming the generalized continuum hypothesis, is each point of $D$ a $2^{1^{st}}$-point for an infinite discrete space $D$?

The author has also been unable to determine if the continuum hypothesis is in fact a necessary condition for Theorem 2.2.

References


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